Chapter 6: Momentum Analysis of Flow Systems

Introduction

- Fluid flow problems can be analyzed using one of three basic approaches: differential, experimental, and integral (or control volume).
- In Chap. 5, control volume forms of the mass and energy equation were developed and used.
- In this chapter, we complete control volume analysis by presenting the integral momentum equation.
  - Review Newton's laws and conservation relations for momentum.
  - Use RTT to develop linear and angular momentum equations for control volumes.
  - Use these equations to determine forces and torques acting on the CV.

Objectives

- After completing this chapter, you should be able to
  - Identify the various kinds of forces and moments acting on a control volume.
  - Use control volume analysis to determine the forces associated with fluid flow.
  - Use control volume analysis to determine the moments caused by fluid flow and the torque transmitted.

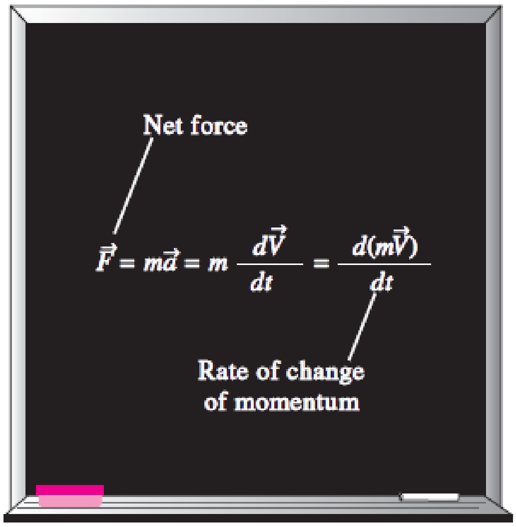
Newton's Laws

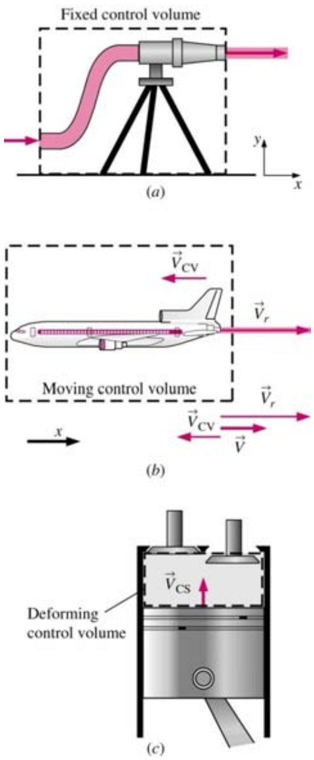
- Newton's laws are relations between motions of bodies and the forces acting on them.
  - **First law**: a body at rest remains at rest, and a body in motion remains in motion at the same velocity in a straight path when the net force acting on it is zero.
  - **Second law**: the acceleration of a body is proportional to the net force acting on it and is inversely proportional to its mass.

$$\vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt} = \frac{d\left(m\vec{V}\right)}{dt}$$

  - **Third law**: when a body exerts a force on a second body, the second body exerts an equal and opposite force on the first.

Newton's Laws





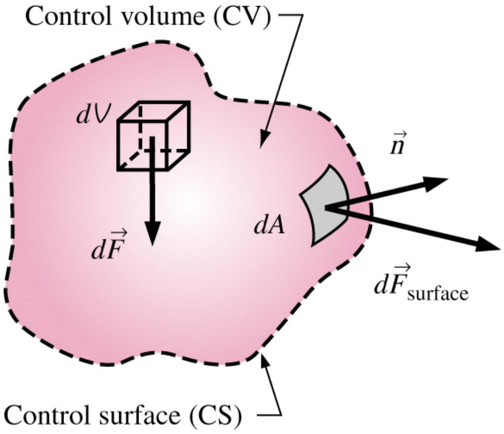
- CV is arbitrarily chosen by fluid dynamicist, however, selection of CV can either simplify or complicate analysis.
  - Clearly define all boundaries. Analysis is often simplified if CS is normal to flow direction.
  - Clearly identify all fluxes crossing the CS.
  - Clearly identify forces and torques *of interest* acting on the CV and CS.
- Fixed, moving, and deforming control volumes.
  - For moving CV, use relative velocity,

$$\vec{V}_r = \vec{V} - \vec{V}_{CV}$$

  - For deforming CV, use relative velocity all deforming control surfaces,

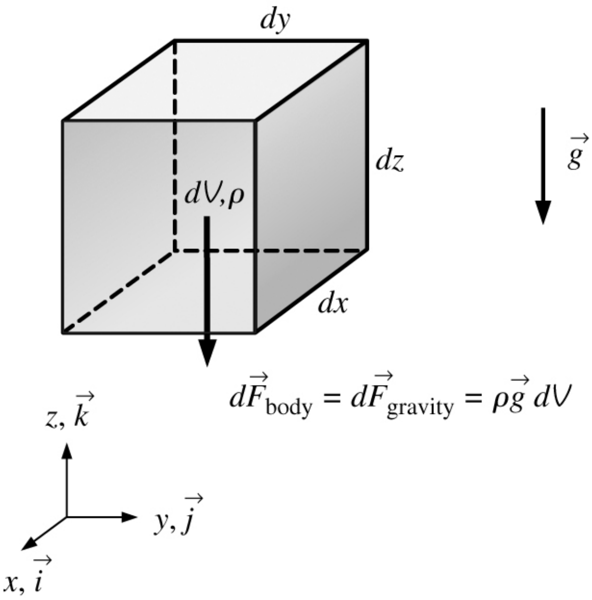
$$\vec{V}_r = \vec{V} - \vec{V}_{CS}$$

- Forces acting on CV consist of **body forces** that act throughout the entire body of the CV (such as gravity, electric, and magnetic forces) and **surface forces** that act on the control surface (such as pressure and viscous forces, and reaction forces at points of contact).



- Body forces act on each volumetric portion dV of the CV.
- Surface forces act on each portion dA of the CS.

Body Forces



- The most common body force is gravity, which exerts a downward force on every differential element of the CV
- The different body force

$$d\vec{F}_{body} = d\vec{F}_{gravity} = \rho\vec{g}d\mathcal{V}$$

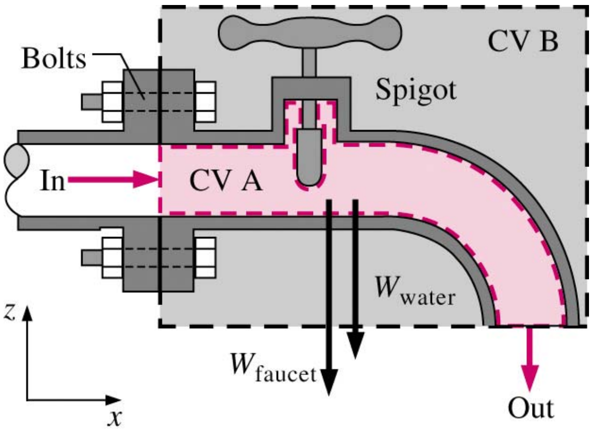
- Typical convention is that acts in the negative z-direction, $\vec{g}$

$$\vec{g} = -g\vec{k}$$

- Total body force acting on CV

$$\sum \vec{F}_{body} = \int_{CV} \rho\vec{g}d\mathcal{V} = m_{CV}\vec{g}$$

Body and Surface Forces



- Surface integrals are cumbersome.
- Careful selection of CV allows expression of total force in terms of more readily available quantities like weight, pressure, and reaction forces.
- Goal is to choose CV to expose only the forces to be determined and a minimum number of other forces.

$$\sum \vec{F} = \underbrace{\sum \vec{F}_{gravity}}_{\text{body force}} + \underbrace{\sum \vec{F}_{pressure} + \sum \vec{F}_{viscous} + \sum \vec{F}_{other}}_{\text{surface forces}}$$

Linear Momentum Equation

- Newton's second law for a system of mass *m* subjected to a force *F* is expressed as

$$\vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt} = \frac{d}{dt}\left(m\vec{V}\right)$$

Linear Momentum Equation

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b \, dV + \int_{CS} \rho b(\vec{V}_r \cdot \vec{n}\,)\, dA$$

$$B = m\vec{V} \qquad b = \vec{V} \qquad b = \vec{V}$$

$$\boxed{\frac{d(m\vec{V})_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho\vec{V}\, dV + \int_{CS} \rho\vec{V}(\vec{V}_r \cdot \vec{n}\,)\, dA}$$

Linear Momentum Equation

The general form of the linear momentum equation that applies to ***fixed, moving, or deforming control volumes*** :

$$\sum \vec{F} = \frac{d}{dt}\int_{cv} \rho\vec{V} d\forall + \int_{cs} \rho\vec{V}(\vec{V}_r \cdot \vec{n}\,)dA$$

$$\begin{pmatrix}\text{The sum of all}\\ \text{external forces}\\ \text{acting on a CV}\end{pmatrix} = \begin{pmatrix}\text{The time rate of change}\\ \text{of the linear momentum}\\ \text{of the contents of the CV}\end{pmatrix} + \begin{pmatrix}\text{The net flow rate of}\\ \text{linear momentum out of the}\\ \text{control surface by mass flow}\end{pmatrix}$$

Fixed CV : $$\sum \vec{F} = \frac{d}{dt}\int_{cv} \rho\vec{V} d\forall + \int_{cs} \rho\vec{V}(\vec{V} \cdot \vec{n}\,)dA$$

Special Cases

$$\sum \vec{F} = \int_{CS} \rho \vec{V} \left(\vec{V}_r \cdot \vec{n} \right) dA$$

- Steady Flow
- Average velocities $\dot{m} = \int_{A_C} \rho \left(\vec{V} \cdot \vec{n} \right) dA_c = \rho V_{avg} A_c$
- Approximate momentum flow rate

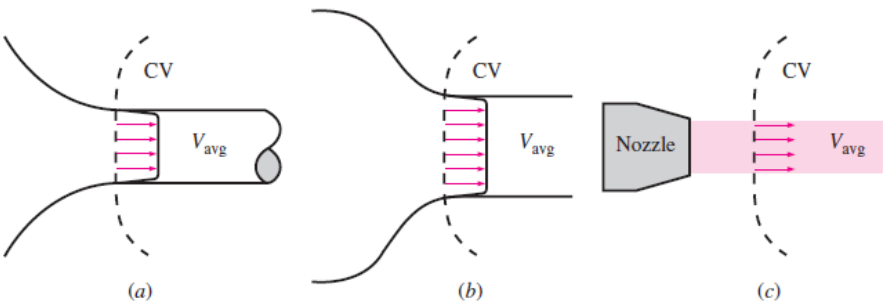
$$\int_{CS} \rho \vec{V} \left(\vec{V} \cdot \vec{n} \right) dA \approx \rho V_{avg} A_c \vec{V}_{avg} = \dot{m} \vec{V}_{avg}$$

- To account for error, use momentum-flux correction factor β

$$\sum \vec{F} = \frac{d}{dt} \int \rho \vec{V} \, d\mathcal{V} + \sum_{out} \beta \dot{m} \vec{V}_{avg} - \sum_{in} \beta \dot{m} \vec{V}_{avg}$$

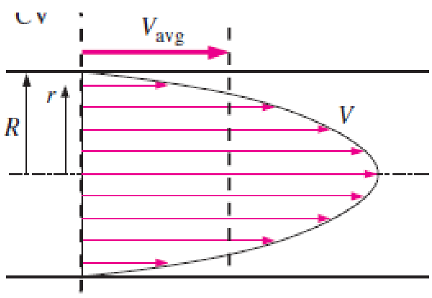
$$\int_{Ac} \rho \vec{V} (\vec{V} \cdot \vec{n}) dA_c = \beta \dot{m} \vec{V}_{avg} \Rightarrow \beta = \frac{1}{A_c} \int_{A_C} \left(\frac{V}{V_{avg}} \right)^2 dA_c$$

Special Cases

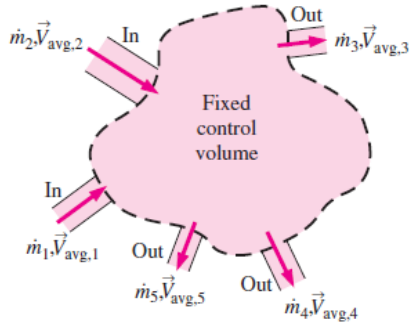


Examples of inlets or outlets in which the uniform flow *(β = 1)* approximation is reasonable: ***(a)* *the well-rounded entrance to* a pipe**, ***(b)* *the entrance to a wind* tunnel test section**, and ***(c)* *a slice* through a free water jet in air.**

β = 4/3 (laminar flow) and 1.01-1.04 (turbulent flow)



Special Cases



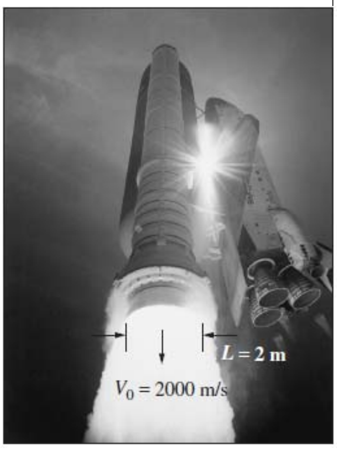
Steady linear momentum equation :

$$\sum \vec{F} = \sum_{out} \beta \dot{m}\vec{V} - \sum_{in} \beta \dot{m}\vec{V}$$

Flow with No External Forces :
$$0 = \frac{d(m\vec{V})_{cv}}{dt} + \sum_{out} \beta \dot{m}\vec{V} - \sum_{in} \beta \dot{m}\vec{V}$$

$$\frac{d(m\vec{V})_{CV}}{dt} = m_{CV}\frac{d\vec{V}_{CV}}{dt} = (m\vec{a})_{CV}$$

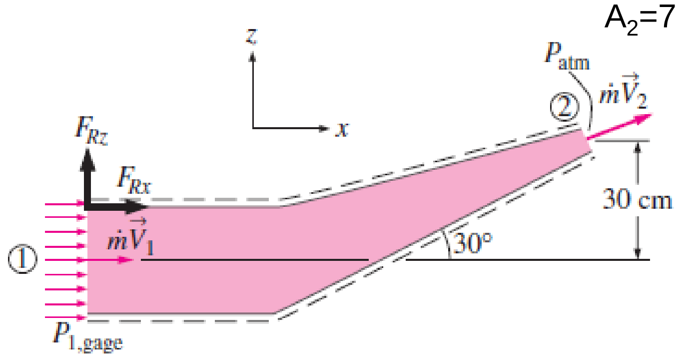
Trust: $$\vec{F}_{body} = m_{body}\vec{a} = \sum_{in} \beta \dot{m}\vec{V} - \sum_{out} \beta \dot{m}\vec{V}$$



The thrust needed to lift the space shuttle is generated by the rocket engines as a result of momentum change of the fuel as it is accelerated from about zero to an exit speed of about 2000 m/s after combustion.

EXAMPLE 6–2 The Force to Hold a Deflector Elbow in Place

A_2=7 cm^2



A_1=113 cm^2

Determine

(a) the gage pressure at the center of the inlet of the elbow ($P_{1,gage}$)

(b) the anchoring force needed to hold the elbow in place ($\mathbf{F_R}$)

EXAMPLE 6–2 The Force to Hold a Deflector Elbow in Place

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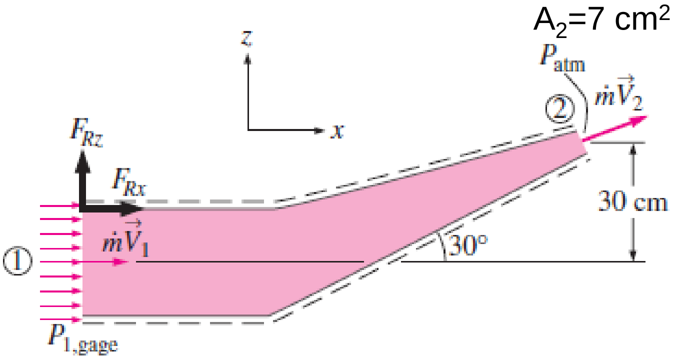


A_1=113 cm^2

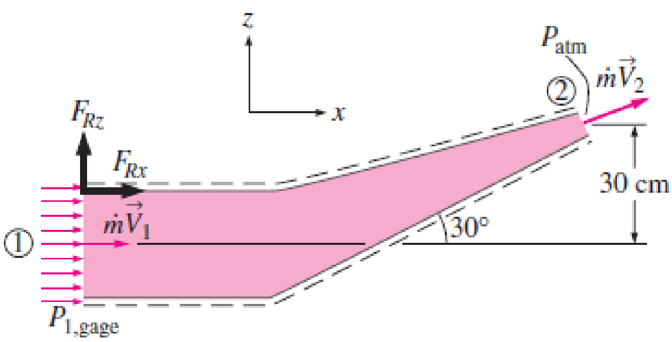
Assumptions :

1) ***The flow is steady, and the frictional effects are negligible.***
2) **The weight of the elbow and the water in it is negligible.**
3) **The water is** discharged to the atmosphere, and thus the gage pressure at the outlet is zero.
4) **The flow is turbulent and fully developed at both the inlet and outlet** of the control volume, and we take the momentum-flux correction factor β to be 1.03.

$$V_1 = \frac{\dot{m}}{\rho A_1} = \frac{14\ \text{kg/s}}{(1000\ \text{kg/m}^3)(0.0113\ \text{m}^2)} = 1.24\ \text{m/s}$$

$$V_2 = \frac{\dot{m}}{\rho A_2} = \frac{14\ \text{kg/s}}{(1000\ \text{kg/m}^3)(7 \times 10^{-4}\ \text{m}^2)} = 20.0\ \text{m/s}$$

EXAMPLE 6–2 The Force to Hold a Deflector Elbow in Place



The Bernoulli equation for a streamline going through the center of the elbow is expressed as

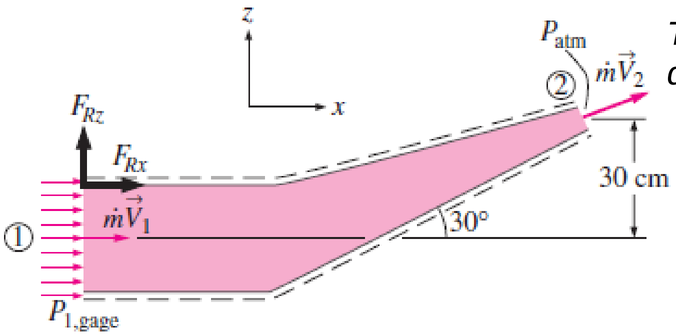
$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

$$P_1 - P_2 = \rho g\left(\frac{V_2^2 - V_1^2}{2g} + z_2 - z_1\right)$$

$$P_1 - P_{atm} = (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2) \times \left(\frac{(20\ \text{m/s})^2 - (1.24\ \text{m/s})^2}{2(9.81\ \text{m/s}^2)} + 0.3 - 0\right)\left(\frac{1\ \text{kN}}{1000\ \text{kg}\cdot\text{m/s}^2}\right)$$

$$\boxed{P_{1,\,\text{gage}} = 202.2\ \text{kN/m}^2 = 202.2\ \text{kPa}\quad(\text{gage})}$$

EXAMPLE 6–2 The Force to Hold a Deflector Elbow in Place



The momentum equation for steady one-dimensional flow is

$$\sum \vec{F} = \sum_{out} \beta \dot{m}\vec{V} - \sum_{in} \beta \dot{m}\vec{V}$$

$$F_{Rx} + P_{1,\,gage}A_1 = \beta \dot{m}V_2 \cos\theta - \beta \dot{m}V_1$$

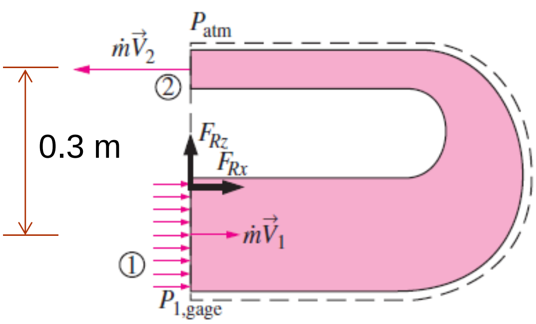
$$F_{Rz} = \beta \dot{m}V_2 \sin\theta$$

$$\begin{aligned} F_{Rx} &= \beta \dot{m}(V_2\cos\theta - V_1) - P_{1,\,gage}A_1 \\ &= 1.03(14\ \text{kg/s})[(20\cos 30^\circ - 1.24)\ \text{m/s}]\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right) \\ &\quad - (202{,}200\ \text{N/m}^2)(0.0113\ \text{m}^2) \\ &= 232 - 2285 = \mathbf{-2053\ N} \end{aligned}$$

$$F_{Rz} = \beta \dot{m}V_2 \sin\theta = (1.03)(14\ \text{kg/s})(20\sin 30^\circ\ \text{m/s})\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right) = \mathbf{144\ N}$$

The negative result for F_{Rx} *indicates that the assumed direction is wrong,* and it should be reversed. Therefore, F_{Rx} *acts in the negative x-direction.*
Discussion :There is a nonzero pressure distribution along the inside walls of the elbow, but since the control volume is outside the elbow, these pressures do not appear in our analysis.

EXAMPLE 6–3 The Force to Hold a Reversing Elbow in Place



Determine the **anchoring force *F_R*** needed to hold the elbow in place.

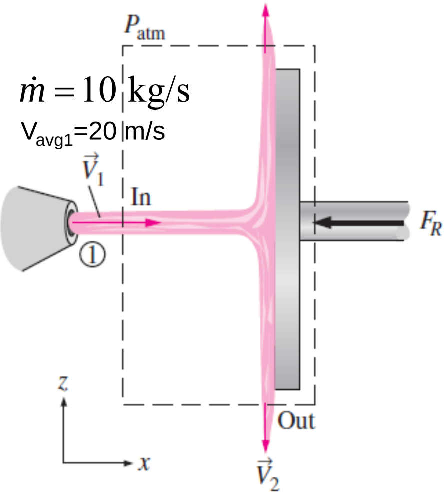
We are neglecting the weight of the elbow and the water.

$$F_{Rx} + P_{1,\,gage}A_1 = \beta_2\dot{m}(-V_2) - \beta_1\dot{m}V_1 = -\beta\dot{m}(V_2 + V_1)$$

$$\begin{aligned} F_{Rx} &= -\beta\dot{m}(V_2 + V_1) - P_{1,\,gage}A_1 \\ &= -(1.03)(14\ \text{kg/s})[(20 + 1.24)\ \text{m/s}]\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right) - (202{,}200\ \text{N/m}^2)(0.0113\ \text{m}^2) \\ &= -306 - 2285 = \mathbf{-2591\ N} \end{aligned}$$

The horizontal force on the flange is 2591 N acting in the negative *x-direction (**the elbow is trying to separate from the pipe**). **This force is*** **equivalent to the weight of about 260 kg mass**, and thus the connectors (such as bolts) used must be strong enough to withstand this force.

EXAMPLE 6–4 Water Jet Striking a Stationary Plate



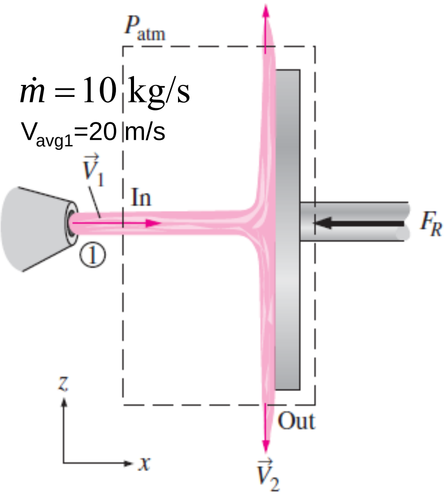
After the strike, the water stream splatters off in all directions in the plane of the plate.

Determine the force needed to prevent the plate from moving horizontally due to the water stream.

Assumptions

1) The water jet is exposed to the atmosphere, and thus the pressure of the water jet and the splattered water leaving the control volume is atmospheric pressure, which is disregarded since it acts on the entire system.
2) **The effect of the momentum-flux** correction factor is negligible, and thus β=*1*.

EXAMPLE 6–4 Water Jet Striking a Stationary Plate



The momentum equation for steady one-dimensional flow

$$\sum \vec{F} = \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$$

$$-F_R = 0 - \beta \dot{m} \vec{V}_1$$

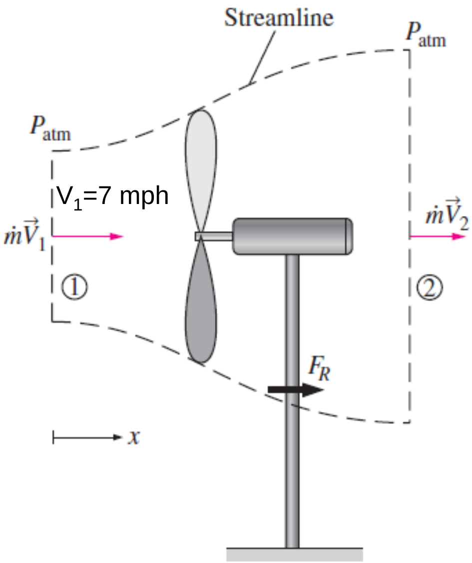
$$F_R = \beta \dot{m} \vec{V}_1 = (1)(10 \text{ kg/s})(20 \text{ m/s})\left(\frac{1 \text{ N}}{1 \text{ kg} \cdot \text{m/s}^2}\right) = \mathbf{200 \text{ N}}$$

(equivalent to the weight of about a 20-kg mass)

Discussion:

- If the control volume were drawn instead along the interface between the water and the plate, there would be additional (unknown) pressure forces in the analysis.
-By cutting the control volume through the support, we avoid having to deal with this additional complexity.
-This is an example of a “wise” choice of control volume.

EXAMPLE 6–5 Power Generation and Wind Loading of a Wind Turbine



A wind generator with a **30-ft-diameter** blade span.

The turbine generates 0.4 kW of electric power.

ρ_{air} = 0.076 lbm/ft^3

Determine:

(a) the efficiency of the wind turbine–generator unit

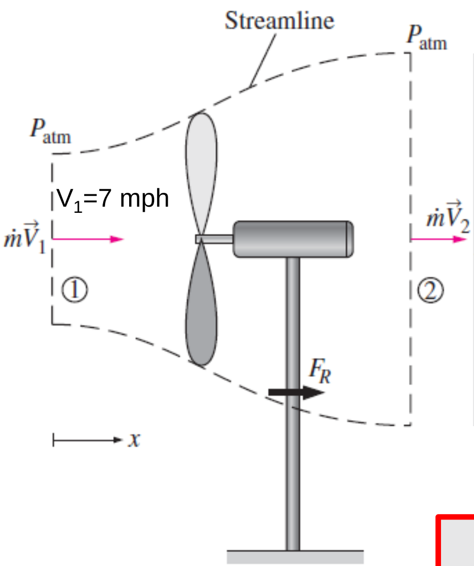
(b) the horizontal force exerted by the wind on the supporting mast of the wind turbine.

(c) What is the effect of doubling the wind velocity to 14 mph on power generation and the force exerted? Assume the efficiency remains the same.

Assumptions

1) ***The efficiency*** of the turbine–generator is independent of wind speed.
2) **The frictional** effects are negligible, and thus none of the incoming kinetic energy is converted to thermal energy.
3) **The average velocity of air through the wind turbine** is the same as the wind velocity (actually, it is considerably less—see the discussion that follows the example).

EXAMPLE 6–5 Power Generation and Wind Loading of a Wind Turbine



The power potential of the wind is proportional to its kinetic energy

$$V_1 = (7\text{ mph})\left(\frac{1.4667\text{ ft/s}}{1\text{ mph}}\right) = 10.27\text{ ft/s}$$

$$\dot{m} = \rho_1 V_1 A_1 = \rho_1 V_1 \frac{\pi D^2}{4} = (0.076\text{ lbm/ft}^3)(10.27\text{ ft/s})\frac{\pi(30\text{ ft})^2}{4} = 551.7\text{ lbm/s}$$

$$\dot{W}_{max} = \dot{m}\text{ke}_1 = \dot{m}\frac{V_1^2}{2}$$

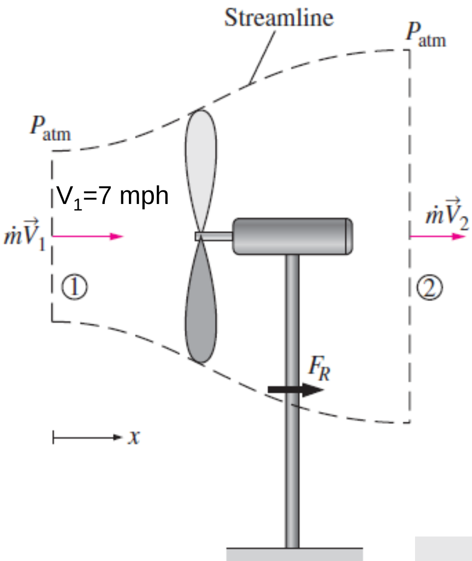
$$= (551.7\text{ lbm/s})\frac{(10.27\text{ ft/s})^2}{2}\left(\frac{1\text{ lbf}}{32.2\text{ lbm}\cdot\text{ft/s}^2}\right)\left(\frac{1\text{ kW}}{737.56\text{ lbf}\cdot\text{ft/s}}\right)$$

$$= 1.225\text{ kW}$$

Then the turbine–generator efficiency becomes :

$$\boxed{\eta_{\text{wind turbine}} = \frac{\dot{W}_{act}}{\dot{W}_{max}} = \frac{0.4\text{ kW}}{1.225\text{ kW}} = 0.327 \qquad (\text{or } 32.7\%)}$$

EXAMPLE 6–5 Power Generation and Wind Loading of a Wind Turbine



The frictional effects are assumed to be negligible, and thus the portion of incoming kinetic energy not converted to electric power leaves the wind turbine as outgoing kinetic energy.

$$\dot{m}\text{ke}_2 = \dot{m}\text{ke}_1(1 - \eta_{\text{wind turbine}}) \quad \rightarrow \quad \dot{m}\frac{V_2^2}{2} = \dot{m}\frac{V_1^2}{2}(1 - \eta_{\text{wind turbine}})$$

$$V_2 = V_1\sqrt{1 - \eta_{\text{wind turbine}}} = (10.27 \text{ ft/s})\sqrt{1 - 0.327} = 8.43 \text{ ft/s}$$

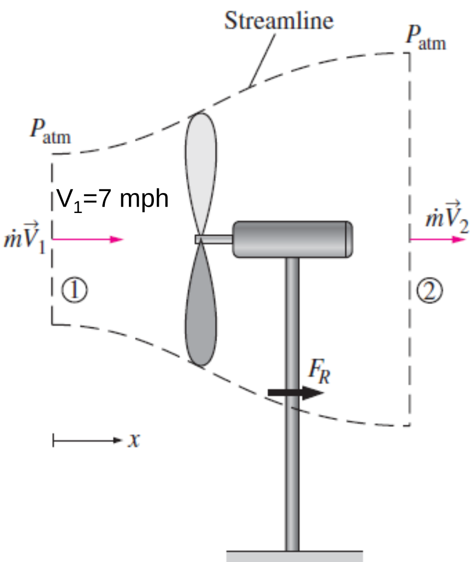
$$\sum \vec{F} = \sum_{\text{out}} \beta\dot{m}\vec{V} - \sum_{\text{in}} \beta\dot{m}\vec{V}$$

$$F_R = \dot{m}V_2 - \dot{m}V_1 = \dot{m}(V_2 - V_1)$$

$$F_R = \dot{m}(V_2 - V_1) = (551.7 \text{ lbm/s})(8.43 - 10.27 \text{ ft/s})\left(\frac{1 \text{ lbf}}{32.2 \text{ lbm} \cdot \text{ft/s}^2}\right)$$
$$= -31.5 \text{ lbf}$$

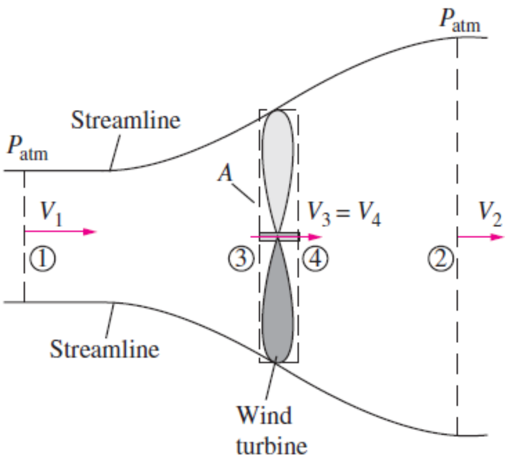
Then the force exerted by the wind on the mast becomes $F_{mast} = -F_R =$ ***31.5 lbf***

EXAMPLE 6–5 Power Generation and Wind Loading of a Wind Turbine



The power generated is proportional to V^3 since the mass flow rate is proportional to V and the kinetic energy to V^2. Therefore, doubling the wind velocity to 14 mph will increase the power generation by a factor of $2^3 = 8$ to $0.4 \times 8 = 3.2$ kW. The force exerted by the wind on the support mast is proportional to V^2. Therefore, doubling the wind velocity to 14 mph will increase the wind force by a factor of $2^2 = 4$ to $31.5 \times 4 = 126$ lbf.

EXAMPLE 6–5 Power Generation and Wind Loading of a Wind Turbine



Discussion:

$$F_R = \dot{m}(V_2 - V_1) \qquad (1)$$

➢The smaller control volume between sections 3 and 4 encloses the turbine, and $A_3 = A_4 = A$ *and* $V_3 = V_4$ *since it is so slim.*

➢*The turbine is a device* that causes a pressure change, and thus the pressures P_3 *and* P_4 *are different.*

➢The momentum equation applied to the smaller control volume gives

$$F_R + P_3A - P_4A = 0 \quad \rightarrow \quad F_R = (P_4 - P_3)A \qquad (2)$$

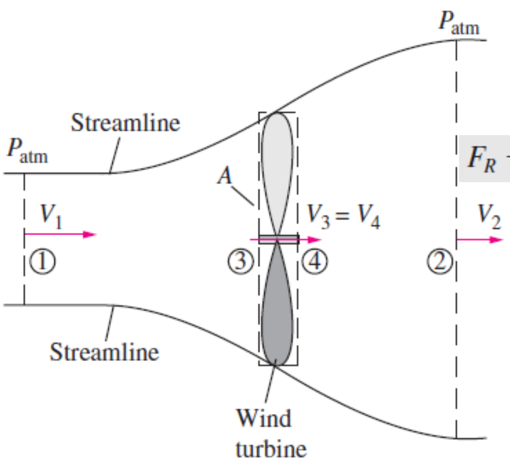
➢ Writing Bernoulli Equation for two section (before and after the rotor blades)

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_3}{\rho g} + \frac{V_3^2}{2g} + z_3 \quad \text{and} \quad \frac{P_4}{\rho g} + \frac{V_4^2}{2g} + z_4 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

Adding these two equations and noting that $z_1 = z_2 = z_3 = z_4$, $V_3 = V_4$, and $P_1 = P_2 = P_{atm}$ gives

$$\frac{V_2^2 - V_1^2}{2} = \frac{P_4 - P_3}{\rho} \qquad (3)$$

EXAMPLE 6–5 Power Generation and Wind Loading of a Wind Turbine



$$F_R = \dot{m}(V_2 - V_1) \qquad (1)$$

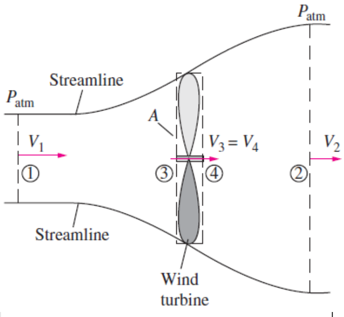
$$F_R + P_3A - P_4A = 0 \quad \rightarrow \quad F_R = (P_4 - P_3)A \qquad (2)$$

$$\frac{V_2^2 - V_1^2}{2} = \frac{P_4 - P_3}{\rho} \qquad (3)$$

Substituting $\dot{m} = \rho A V_3$ into Eq. 1 and then combining it with Eqs. 2 and 3 gives

$$V_3 = \frac{V_1 + V_2}{2} \qquad (4)$$

EXAMPLE 6–5 Power Generation and Wind Loading of a Wind Turbine



Now back to the wind turbine. The velocity through the turbine can be expressed as $V_3 = V_1(1 - a)$, where $a < 1$ since $V_3 < V_1$. Combining this expression with Eq. 4 gives $V_2 = V_1(1 - 2a)$. Also, the mass flow rate through the turbine becomes $\dot{m} = \rho A V_3 = \rho A V_1(1 - a)$. When the frictional effects and losses are neglected, the power generated by a wind turbine is simply the difference between the incoming and the outgoing kinetic energies:

$$\dot{W} = \dot{m}(\mathrm{ke}_1 - \mathrm{ke}_2) = \frac{\dot{m}(V_1^2 - V_2^2)}{2} = \frac{\rho A V_1(1-a)[V_1^2 - V_1^2(1-2a)^2]}{2}$$

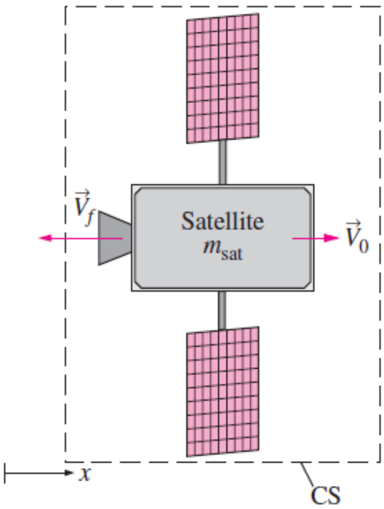
$$= 2\rho A V_1^3 a(1-a)^2$$

Dividing this by the available power of the wind $\dot{W}_{max} = \dot{m}V_1^2/2$ gives the efficiency of the wind turbine in terms of a,

$$\eta_{\text{wind turbine}} = \frac{\dot{W}}{\dot{W}_{max}} = \frac{2\rho A V_1^3 a(1-a)^2}{(\rho A V_1)V_1^2/2}$$

The value of a that maximizes the efficiency is determined by setting the derivative of $\eta_{\text{wind turbine}}$ with respect to a equal to zero and solving for a. It gives $a = 1/3$. Substituting this value into the efficiency relation just presented gives $\eta_{\text{wind turbine}} = 16/27 = 0.593$, which is the upper limit for the efficiency of wind turbines and propellers. This is known as the **Betz limit**. The efficiency of actual wind turbines is about half of this ideal value.

EXAMPLE 6–6 Repositioning of a Orbiting Satellite



An orbiting satellite has a mass of m_{sat} = *5000 kg and is traveling at a constant* velocity of V_0

An attached rocket discharges m_f=100 kg of gases from the reaction of solid fuel at a velocity V_f *=3000 m/s* relative to the satellite.

The fuel discharge rate is constant for 2 s.

<u>Determine :</u>

(a) the acceleration of the satellite during this 2-s period

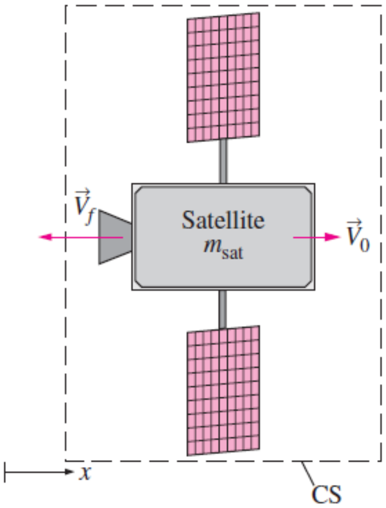
(b) the change of velocity of the satellite during this time period

(c) the thrust exerted on the satellite

<u>Assumptions :</u>

1) **There are no external forces acting on the satellite**
2) The effect of the pressure force at the nozzle exit is negligible.
3) **The mass of discharged fuel is negligible relative to the mass of the satellite**

EXAMPLE 6–6 Repositioning of a Orbiting Satellite



The satellite can be treated as a solid body with constant mass

$$0 = \frac{d(m\vec{V})_{CV}}{dt} + \sum_{out} \beta \dot{m}\vec{V} - \sum_{in} \beta \dot{m}\vec{V} \quad \rightarrow \quad m_{sat}\frac{d\vec{V}_{sat}}{dt} = -\dot{m}_f \vec{V}_f$$

$$m_{sat}\frac{dV_{sat}}{dt} = \dot{m}_f V_f \quad \rightarrow \quad \frac{dV_{sat}}{dt} = \frac{\dot{m}_f}{m_{sat}} V_f = \frac{m_f/\Delta t}{m_{sat}} V_f$$

$$a_{sat} = \frac{dV_{sat}}{dt} = \frac{m_f/\Delta t}{m_{sat}} V_f = \frac{(100 \text{ kg})/(2 \text{ s})}{5000 \text{ kg}}(3000 \text{ m/s}) = \mathbf{30 \text{ m/s}^2} \quad \text{(a)}$$

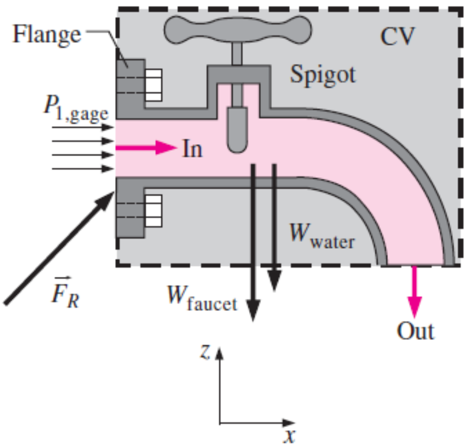
- The velocity change of the satellite during the first 2 s

$$dV_{sat} = a_{sat}\, dt \quad \rightarrow \quad \Delta V_{sat} = a_{sat}\, \Delta t = (30 \text{ m/s}^2)(2 \text{ s}) = \mathbf{60 \text{ m/s}} \quad \text{(b)}$$

- The thrust exerted on the satellite is

$$F_{sat} = 0 - \dot{m}_f(-V_f) = -(100/2 \text{ kg/s})(-3000 \text{ m/s})\left(\frac{1 \text{ kN}}{1000 \text{ kg} \cdot \text{m/s}^2}\right) = \mathbf{150 \text{ kN}} \quad \text{(c)}$$

EXAMPLE 6–7 Net Force on a Flange



Flow rate, Q = 18.5 gal/min

Density of water, ρ = 62.3 lbm/ft^3

The inner diameter of the pipe, D = 0.065 ft

$P_{1,gage}$ = 13.0 psig

$W_{water} + W_{faucet}$ = 12.8 lbf

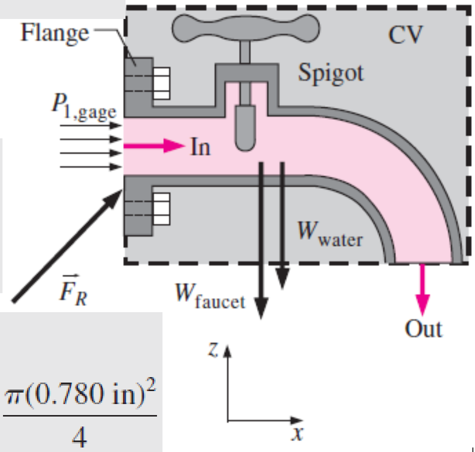
Calculate the net force on the flange.

Assumption: The flow at the inlet and at the outlet is turbulent and fully developed so that the Momentum flux correction factor is about 1.03.

EXAMPLE 6–7 Net Force on a Flange

$$V_2 = V_1 = V = \frac{\dot{V}}{A_c} = \frac{\dot{V}}{\pi D^2/4} = \frac{18.5 \text{ gal/min}}{\pi(0.065 \text{ ft})^2/4}\left(\frac{0.1337 \text{ ft}^3}{1 \text{ gal}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 12.42 \text{ ft/s}$$

$$\dot{m} = \rho\dot{V} = (62.3 \text{ lbm/ft}^3)(18.5 \text{ gal/min})\left(\frac{0.1337 \text{ ft}^3}{1 \text{ gal}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 2.568 \text{ lbm/s}$$



$$\sum \vec{F} = \sum_{\text{out}} \beta\dot{m}\vec{V} - \sum_{\text{in}} \beta\dot{m}\vec{V}$$

The momentum equations along the *x*- and *z*-directions become

$$F_{Rx} + P_{1,\text{ gage}}A_1 = 0 - \dot{m}(+V_1)$$

$$F_{Rz} - W_{\text{faucet}} - W_{\text{water}} = \dot{m}(-V_2) - 0$$

$$F_{Rx} = -\dot{m}V_1 - P_{1,\text{ gage}}A_1$$

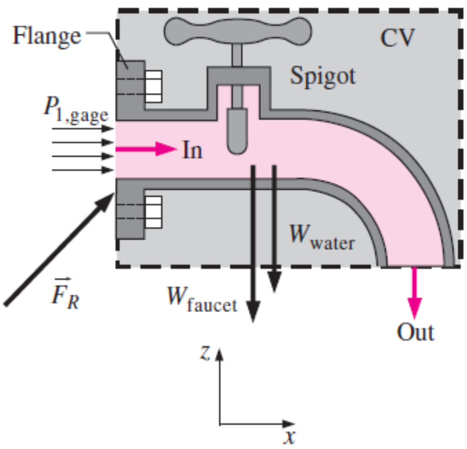
$$= -(2.568 \text{ lbm/s})(12.42 \text{ ft/s})\left(\frac{1 \text{ lbf}}{32.2 \text{ lbm}\cdot\text{ft/s}^2}\right) - (13 \text{ lbf/in}^2)\frac{\pi(0.780 \text{ in})^2}{4}$$

$$= -7.20 \text{ lbf}$$

$$F_{Rz} = -\dot{m}V_2 + W_{\text{faucet+water}}$$

$$= -(2.568 \text{ lbm/s})(12.42 \text{ ft/s})\left(\frac{1 \text{ lbf}}{32.2 \text{ lbm}\cdot\text{ft/s}^2}\right) + 12.8 \text{ lbf} = 11.8 \text{ lbf}$$

EXAMPLE 6–7 Net Force on a Flange



Then the net force of the flange on the control volume

$$\vec{F}_R = F_{Rx}\vec{i} + F_{Rz}\vec{k} = -7.20\vec{i} + 11.8\vec{k} \quad \text{lbf}$$

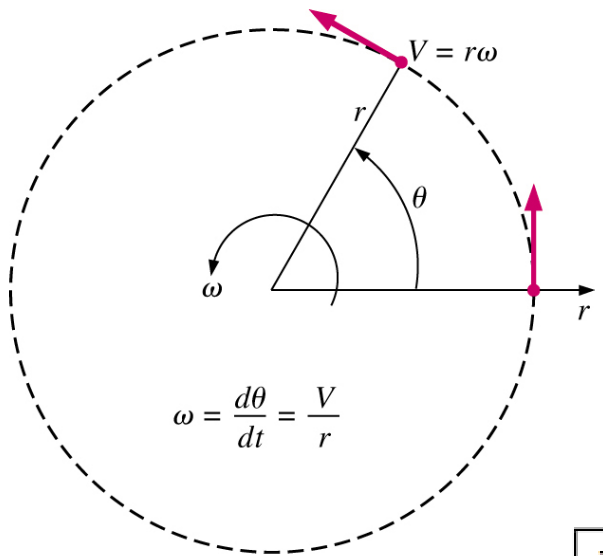
From Newton's third law, the force the faucet assembly exerts on the flange is the negative of $\vec{F}_R$,

$$\vec{F}_{\text{faucet on flange}} = -\vec{F}_R = \mathbf{7.20\vec{i} - 11.8\vec{k}} \quad \mathbf{lbf}$$

Angular Momentum

- Motion of a rigid body can be considered to be the combination of
  - the translational motion of its center of mass (U_x, U_y, U_z)
  - the rotational motion about its center of mass (ω_x, ω_y, ω_z)
- Translational motion can be analyzed with linear momentum equation.
- Rotational motion is analyzed with angular momentum equation.

Review of Rotational Motion



Angular velocity ω is the angular distance θ traveled per unit time, and angular acceleration α is the rate of change of angular velocity.

$$\omega = \frac{d\theta}{dt} = \frac{d(lr)}{dt} = \frac{1}{r}\frac{dl}{dt} = \frac{V}{r}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \frac{1}{r}\frac{dV}{dt} = \frac{a_t}{r}$$

$$\boxed{V = r\omega \text{ and } a_t = r\alpha}$$

Review of Rotational Motion

The torque *M acting on a point mass m* *at a normal distance r from the axis of rotation is expressed as*

$$M = rF_t = rma_t = mr^2\alpha$$

The total torque acting on a rotating rigid body about an axis can be determined by integrating the torques acting on differential masses *dm over the* entire body to give

$$M = \int_{mass} r^2 \alpha dm = \left[\int_{mass} r^2 dm\right]\alpha = I\alpha$$

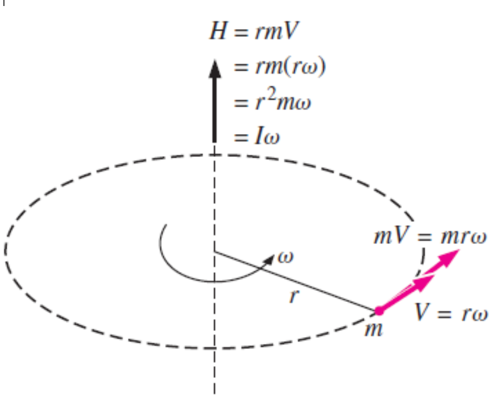
I : The moment of inertia of the body about the axis of rotation (measure of the inertia of a body against rotation)

Review of Rotational Motion

The moment of momentum, called **the angular momentum**, **of a point mass m** about an axis can be expressed as H = rmV = $r^2 m\omega$

Then the total angular momentum of a rotating rigid body can be determined by integration to be

Angular momentum (moment of momentum): $$H = \int_{mass} r^2 \omega dm = \left[\int_{mass} r^2 dm\right]\omega = I\omega$$



Newton's second law : $\vec{F} = m\vec{a}$

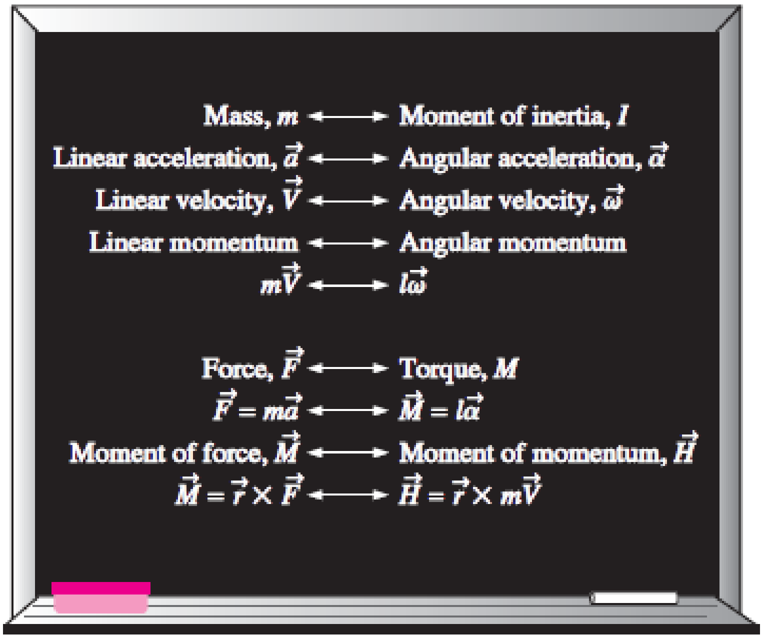
In terms of the rate of change of linear momentum : $\vec{F} = \dfrac{d\left(m\vec{V}\right)}{dt}$

The counterpart of Newton's second law for rotating bodies: $\vec{M} = I\vec{\alpha}$

In terms of the rate of change of angular momentum :

Angular momentum equation: $\vec{M} = \mathrm{I}\vec{\alpha} = \mathrm{I}\dfrac{d\vec{\omega}}{dt} = \dfrac{d(\mathrm{I}\vec{\omega})}{dt} = \dfrac{d\vec{H}}{dt}$

Review of Rotational Motion



Analogy between corresponding
linear and angular quantities

Review of Angular Momentum

- Moment of a force: $\vec{M} = \vec{r} \times \vec{F}$
- Moment of momentum: $\vec{H} = \vec{r} \times m\vec{V}$
- For a system: $\vec{H}_{sys} = \int_{sys} (\vec{r} \times \vec{V})\rho \, d\mathcal{V}$

$$\frac{d\vec{H}_{sys}}{dt} = \frac{d}{dt}\int_{sys} (\vec{r} \times \vec{V})\rho \, d\mathcal{V}$$

- Therefore, the angular momentum equation can be written as: $\sum \vec{M} = \dfrac{d\vec{H}_{sys}}{dt}$

$\sum \vec{M}$: the net torque applied on the system, which is the vector sum of the moments of all forces acting on the system

$\dfrac{d\vec{H}_{sys}}{dt}$: *the rate of change of the angular momentum of the system*

The Angular Momentum Equation

Reynolds Transport Theorem (RTT) :

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b \, dV + \int_{CS} \rho b (V_r \cdot \vec{n})\, dA$$

$$B = \vec{H} \qquad b = \vec{r} \times \vec{V} \qquad b = \vec{r} \times \vec{V}$$

$$\boxed{\frac{dH_{sys}}{dt} = \frac{d}{dt}\int_{CV} (\vec{r} \times \vec{V})\rho \, dV + \int_{CS} (\vec{r} \times \vec{V})\rho(\vec{V}_r \cdot \vec{n})\, dA}$$

(General form)

$$\left(\begin{array}{c}\text{The sum of all}\\ \text{external moments}\\ \text{acting on a CV}\end{array}\right) = \left(\begin{array}{c}\text{The time rate of change}\\ \text{of the angular momentum}\\ \text{of the contents of the CV}\end{array}\right) + \left(\begin{array}{c}\text{The net flow rate of}\\ \text{angular momentum}\\ \text{out of the control}\\ \text{surface by mass flow}\end{array}\right)$$

The Angular Momentum Equation

General form:

$$\sum \vec{M} = \frac{d}{dt}\int_{cv} (\vec{r} \times \vec{V})\rho \, d\forall + \int_{cs} (\vec{r} \times \vec{V})\rho \, (\vec{V}_r . \vec{n}) dA$$

Fixed CV:

$$\sum \vec{M} = \frac{d}{dt}\int_{cv} (\vec{r} \times \vec{V})\rho \, d\forall + \int_{cs} (\vec{r} \times \vec{V})\rho \, (\vec{V} . \vec{n}) dA$$

Steady flow:

$$\sum \vec{M} = \int_{CS} (\vec{r} \times \vec{V})\rho (\vec{V}_r \dot{n}) dA$$

➢ **Approximate form using average properties at inlets and outlets :**

$$\sum \vec{M} = \frac{d}{dt}\int_{CV} (\vec{r} \times \vec{V})\rho \, dV + \sum_{out} \vec{r} \times \dot{m}\vec{V} - \sum_{in} \vec{r} \times \dot{m}\vec{V}$$

Steady flow:

$$\sum \vec{M} = \sum_{out} \vec{r} \times \dot{m}\vec{V} - \sum_{in} \vec{r} \times \dot{m}\vec{V}$$

The Angular Momentum Equation

Flow with No External Moments : $0 = \frac{dH_{cv}}{dt} + \sum_{out} \vec{r} \times \dot{m}\vec{V} - \sum_{in} \vec{r} \times \dot{m}\vec{V}$

This is an expression of the conservation of angular momentum principle, which can be stated as ***in the absence of external moments***, *the rate of change of the angular momentum of a control volume is equal to the difference between the incoming and outgoing angular momentum fluxes.*

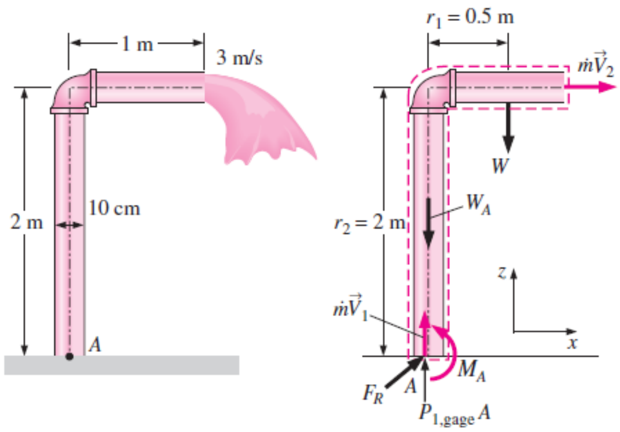
When the moment of inertia *I of the control volume remains constant, the* **first term of the above** equation simply becomes moment of inertia times angular acceleration, $I\vec{\alpha}$

Therefore, the control volume in this case can be **treated as a solid body**, with a net torque of

$$\vec{M}_{body} = I_{body}\vec{\alpha} = \sum_{in}(\vec{r} \times \dot{m}\vec{V}) - \sum_{out}(\vec{r} \times \dot{m}\vec{V})$$

This approach can be used to determine the angular acceleration of space vehicles and aircraft when a rocket is fired in a direction different than the direction of motion.

EXAMPLE 6–8 Bending Moment Acting at the Base of a Water Pipe



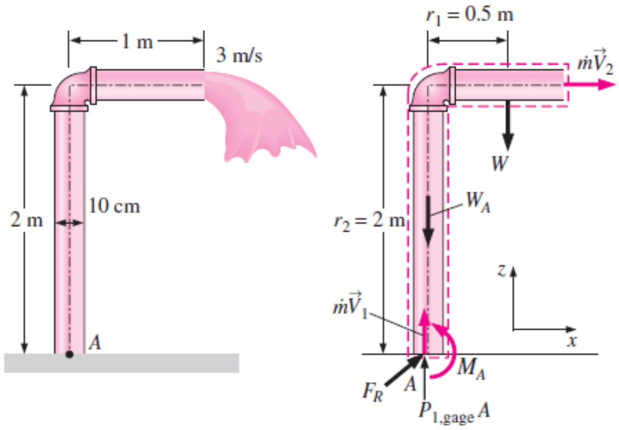
The mass of the horizontal pipe section when filled with water is **12 kg per meter length**

Assumption:

The pipe diameter is small compared to the moment arm, and thus we use average values of radius and velocity at the outlet.

Determine the bending moment acting at the base of the pipe (point *A) and the required length of the horizontal section* that would make the moment at point *A zero.*

EXAMPLE 6–8 Bending Moment Acting at the Base of a Water Pipe



$$\dot{m} = \rho A_c V = (1000 \text{kg/m}^3)[\pi(0{,}10\text{m})^2 / 4](3\text{m/s}) = 23{,}56 \text{ kg/s}$$

$$W = mg = (12\text{kg/m})(1\text{m})(9{,}81\text{m/s}^2)\left(\frac{1\text{ N}}{1\text{ kg}\cdot\text{m/s}^2}\right) = 118 \text{ N}$$

$$\sum M = \sum_{out} r\dot{m}V - \sum_{in} r\dot{m}V$$

$$M_A - r_1 W = -r_2 \dot{m} V_2$$

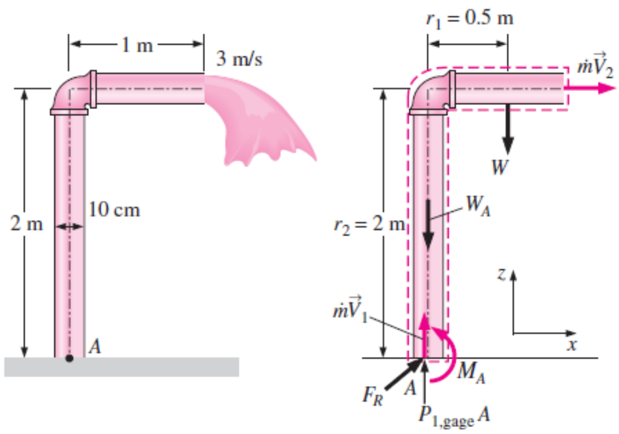
$$M_A = r_1 W - r_2 \dot{m} V_2$$

$$= (0{,}5\text{ m})(118\text{ N}) - (2\text{ m})\ (23{,}56\text{ kg/s})\ (3\text{m/s})\left(\frac{1\text{N}}{1\text{ kg}\cdot\text{m/s}^2}\right)$$

$$= -82.5\text{ N}\cdot\text{ m}$$

The negative sign indicates that the assumed direction for M_A *is wrong and* should be reversed. Therefore, a moment of 82.5 Nm acts at the stem of the pipe in the clockwise direction.

EXAMPLE 6–8 Bending Moment Acting at the Base of a Water Pipe

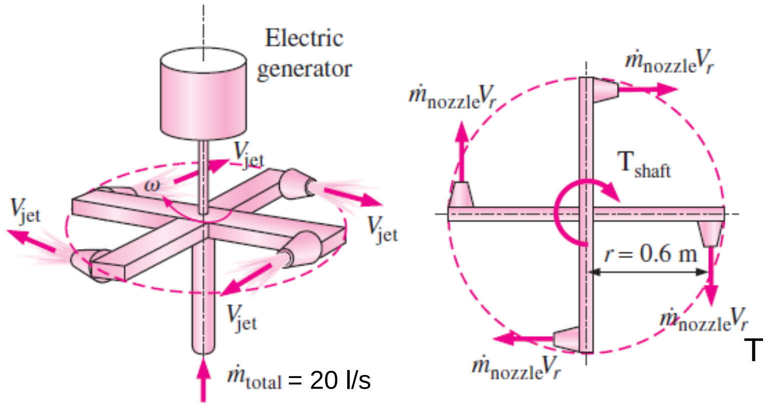


The length *L of the horizontal pipe* that will cause the moment at the pipe stem to vanish is determined to be

$$0 = r_1 W - r_2 \dot{m} V_2 \quad \rightarrow \quad 0 = (L/2)Lw - r_2 \dot{m} V_2$$

$$L = \sqrt{\frac{r_2 \dot{m} V_2}{w}} = \sqrt{\frac{2 \times 141.4\text{ N}\cdot\text{m}}{118\text{ N/m}}} = 2.40\text{ m}$$

EXAMPLE 6–9 Power Generation from a Sprinkler System



A large lawn sprinkler with four identical arms is to be converted into a turbine to generate electric power by attaching a generator to its rotating head.

$$\dot{n} = 300 \text{ rpm}$$

The diameter of each jet is 1 cm.

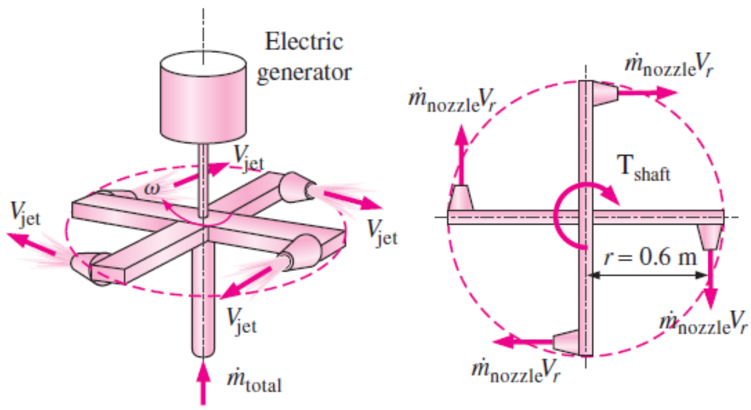
Estimate the electric power produced.

Assumptions :

1) The flow is cyclically steady (i.e., steady from a frame of reference rotating with the sprinkler head).

2) Generator losses and air drag of rotating components are neglected.

EXAMPLE 6–9 Power Generation from a Sprinkler System



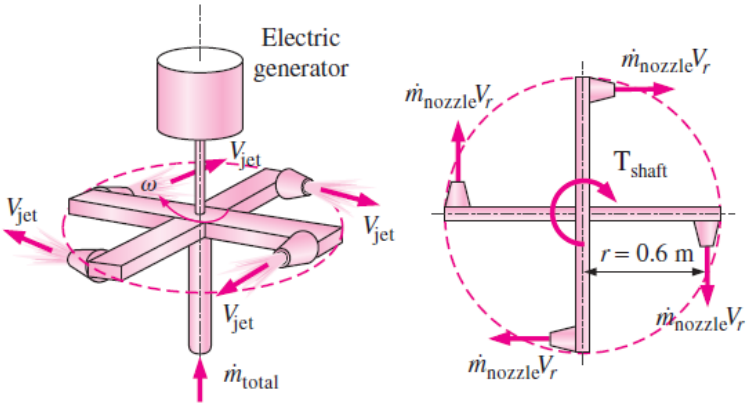
$$V_{jet} = \frac{\dot{\forall}_{nozzle}}{A_{jet}} = \frac{5 \text{ L/s}}{[\pi\,(0{,}01\text{ m})^2/4]}\left(\frac{1\text{m}^3}{1000\text{ L}}\right) = 63{,}66 \text{ m/s}$$

The angular and tangential velocities of the nozzles are

$$\omega = 2\pi\dot{n} = 2\pi(300 \text{ rev/min})\left(\frac{1\text{min}}{60\text{s}}\right) = 31.42 \text{ rad/s}$$

$$V_{nozzle} = r\omega = (0.6 \text{ m})(31.42 \text{ rad/s}) = 18.85 \text{ m/s}$$

EXAMPLE 6–9 Power Generation from a Sprinkler System



Then the average velocity of the water jet relative to the control volume (or relative to a fixed location on earth) becomes

$$V_r = V_{jet} - V_{nozzle} = 63.66 - 18.85 = 44.81 \text{ m/s}$$

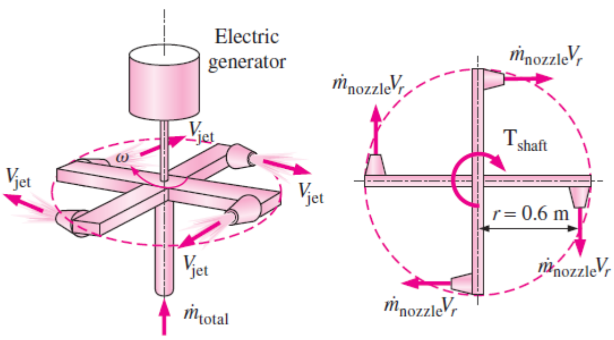
The angular momentum equation about the axis of rotation :

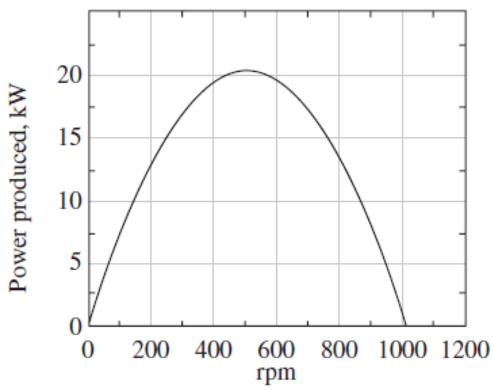
$$-T_{shaft} = -4r\dot{m}_{nozzle}V_r \qquad \text{or} \qquad T_{shaft} = r\dot{m}_{total}V_r$$

$$T_{shaft} = r\dot{m}_{total}V_r = (0{,}6\text{ m})(20\text{ kg/s})(44.81\text{ m/s})\left(\frac{1\text{N}}{1\text{ kg m/s}^2}\right) = 537.7\text{ N}\cdot\text{m}$$

$$\dot{m}_{total} = \rho\dot{\forall}_{total} = (1\text{ kg/L})(20\text{ L/s}) = 20\text{ kg/s}$$

EXAMPLE 6–9 Power Generation from a Sprinkler System





Then the power generated becomes :

$$\dot{W} = 2\pi\dot{n}T_{shaft} = \omega T_{shaft} = (31.42\text{ rad/s})\ (537.7\text{ N}\cdot\text{m})\left(\frac{1\text{ kW}}{1000\text{ N}\cdot\text{m/s}}\right) = 16.9\text{ kW}$$

Discussion:

1) The sprinkler is stuck and thus the angular velocity is zero. The torque developed will be maximum in this case since $V_{nozzle} = 0$ *and thus* $V_r = V_{jet} = 63.66$ *m/s, giving* $T_{shaft,\,max} = 764\ N \cdot m$*. But* the power generated will be zero since the shaft does not rotate.

2) The shaft is disconnected from the generator (and thus both the torque and power generation are zero) and the shaft accelerates until it reaches an equilibrium velocity. Setting $T_{shaft} = 0$ in the angular momentum equation gives $V_r = 0$ *and thus* $V_{jet} = V_{nozzle} = 63.66$ *m/s.* The corresponding angular speed of the sprinkler is

$$\dot{\eta} = \frac{\omega}{2\pi} = \frac{V_{nozzle}}{2\pi r} = \frac{63.66\text{ m/s}}{2\pi\,(0{,}6\text{ m})}\left(\frac{60\text{ s}}{1\text{ min}}\right) = 1013\text{ rpm}$$

Homework

18,21,22,25,27,29,33,36,40, 42,43,47,51,53,58,60,66,73