Fluid Mechanics: Fundamentals and Applications, 2nd Edition
Yunus A. Cengel, John M. Cimbala
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Chapter 6
MOMENTUM ANALYSIS OF FLOW SYSTEMS

Lecture slides by

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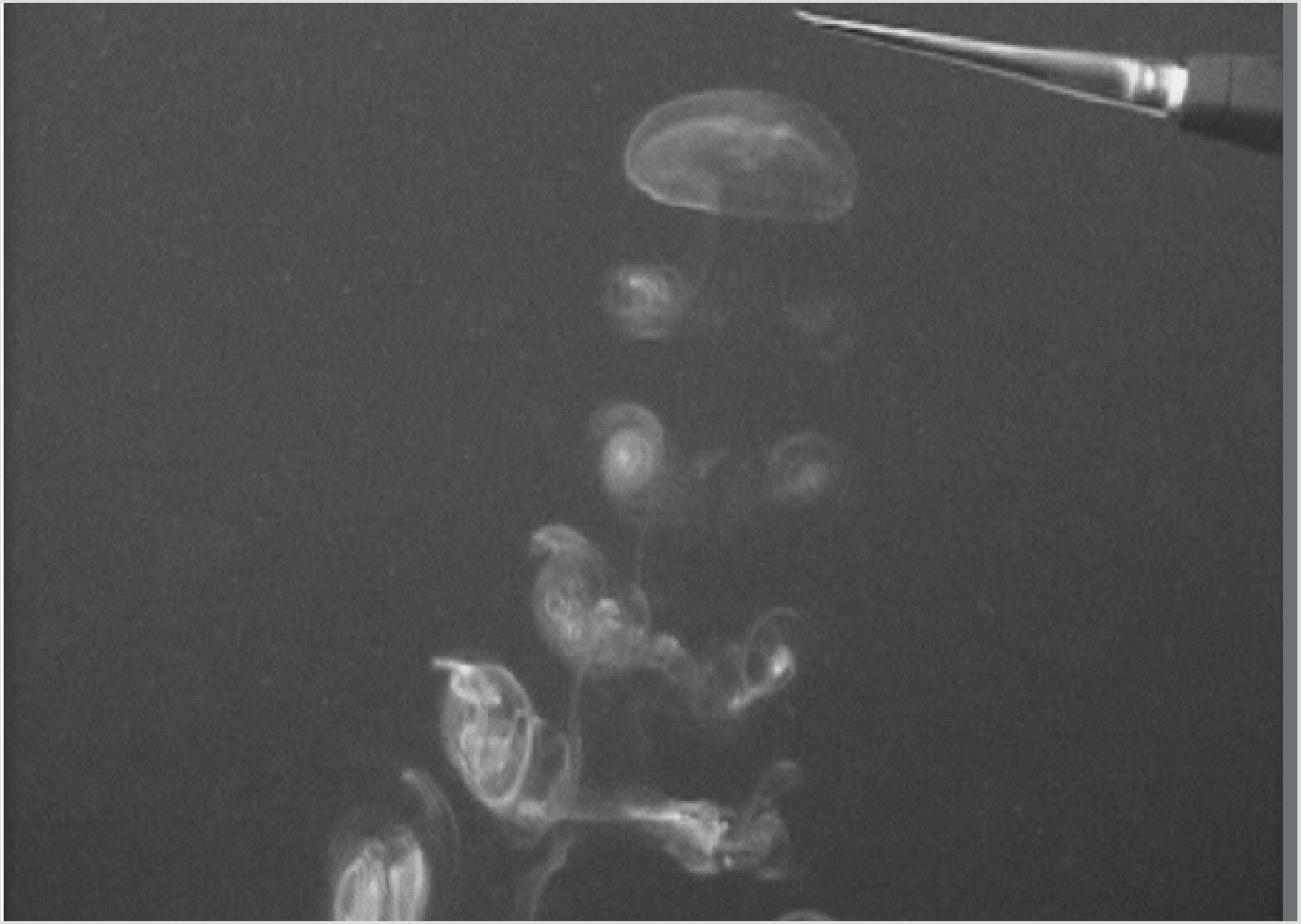


Steady swimming of the jellyfish Aurelia aurita. Fluorescent dye placed directly upstream of the animal is drawn underneath the bell as the body relaxes and forms vortex rings below the animal as the body contracts and ejects fluid. The vortex rings simultaneously induce flows for both feeding and propulsion.

Objectives

- Identify the various kinds of forces and moments acting on a control volume
- Use control volume analysis to determine the forces associated with fluid flow
- Use control volume analysis to determine the moments caused by fluid flow and the torque transmitted

6–1 ■ NEWTON'S LAWS

Newton's laws: Relations between motions of bodies and the forces acting on them.

Newton's first law: A body at rest remains at rest, and a body in motion remains in motion at the same velocity in a straight path when the net force acting on it is zero.

Therefore, a body tends to preserve its state of inertia.

Newton's second law: The acceleration of a body is proportional to the net force acting on it and is inversely proportional to its mass.

Newton's third law: When a body exerts a force on a second body, the second body exerts an equal and opposite force on the first.

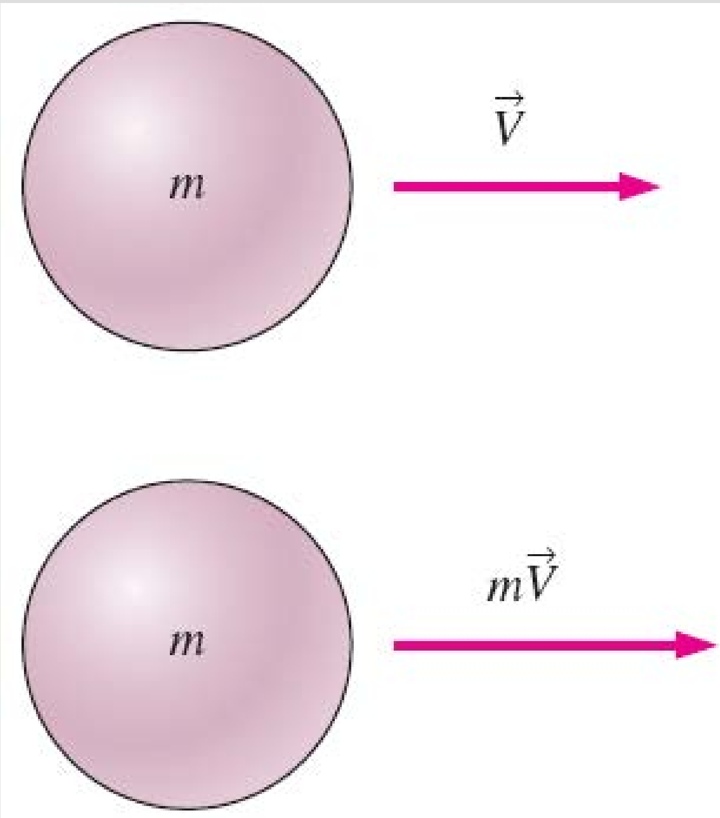
Therefore, the direction of an exposed reaction force depends on the body taken as the system.

Newton's second law: $$\vec{F} = m\vec{a} = m\,\frac{d\vec{V}}{dt} = \frac{d(m\vec{V})}{dt}$$

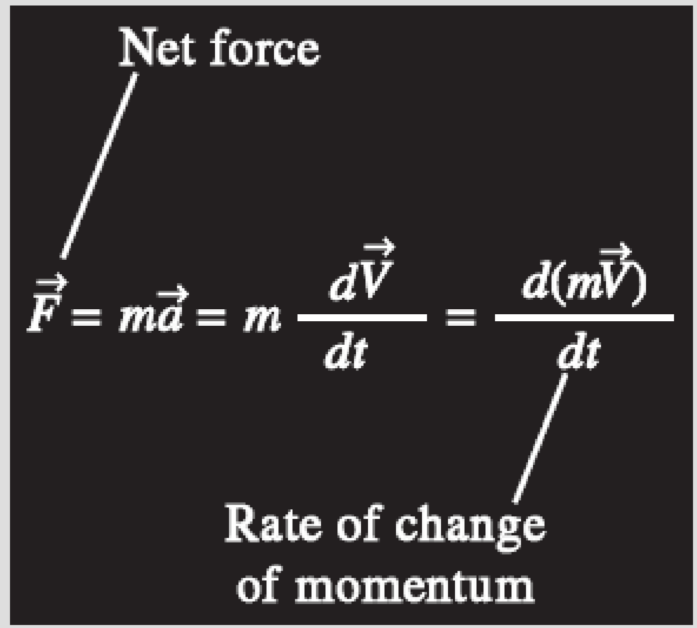
Linear momentum or just the *momentum* of the body:
The product of the mass and the velocity of a body.

Newton's second law is usually referred to as the *linear momentum equation*.

Conservation of momentum principle: The momentum of a system remains constant only when the net force acting on it is zero.



Linear momentum is the product of mass and velocity, and its direction is the direction of velocity.



Newton's second law is also expressed as *the rate of change of the momentum of a body is equal to the net force acting on it*.

The counterpart of Newton's second law for rotating rigid bodies is expressed as $\vec{M} = I\vec{\alpha}$, where $\vec{M}$ is the net moment or torque applied on the body, I is the moment of inertia of the body about the axis of rotation, and $\vec{\alpha}$ is the angular acceleration. It can also be expressed in terms of the rate of change of angular momentum $d\vec{H}/dt$ as

Angular momentum equation:

$$\vec{M} = I\vec{\alpha} = I\frac{d\vec{\omega}}{dt} = \frac{d(I\vec{\omega})}{dt} = \frac{d\vec{H}}{dt} \quad \textbf{(6–2)}$$

Angular momentum about x-axis:

$$M_x = I_x\frac{d\omega_x}{dt} = \frac{dH_x}{dt}$$

The conservation of angular momentum Principle: The total angular momentum of a rotating body remains constant when the net torque acting on it is zero, and thus the angular momentum of such systems is conserved.

Net torque

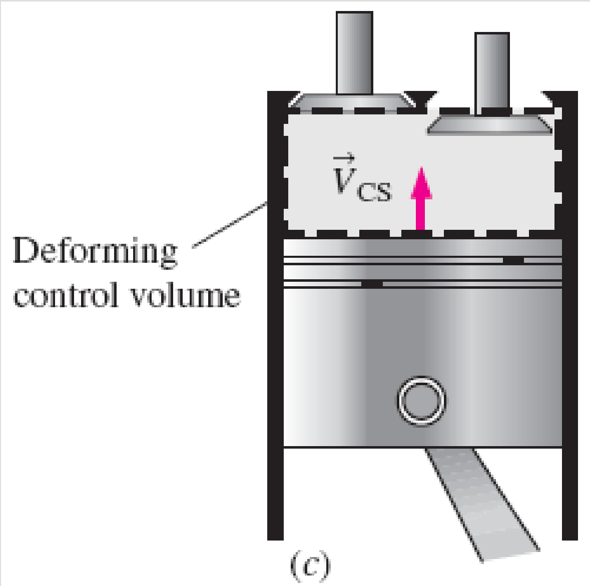
$$\vec{M} = I\vec{\alpha} = I\frac{d\vec{\omega}}{dt} = \frac{d(I\vec{\omega})}{dt} = \frac{d\vec{H}}{dt}$$

Rate of change of angular momentum

The rate of change of the angular momentum of a body is equal to the net torque acting on it.

6–2 ■ CHOOSING A CONTROL VOLUME

A control volume can be selected as any arbitrary region in space through which fluid flows, and its bounding control surface can be fixed, moving, and even deforming during flow.

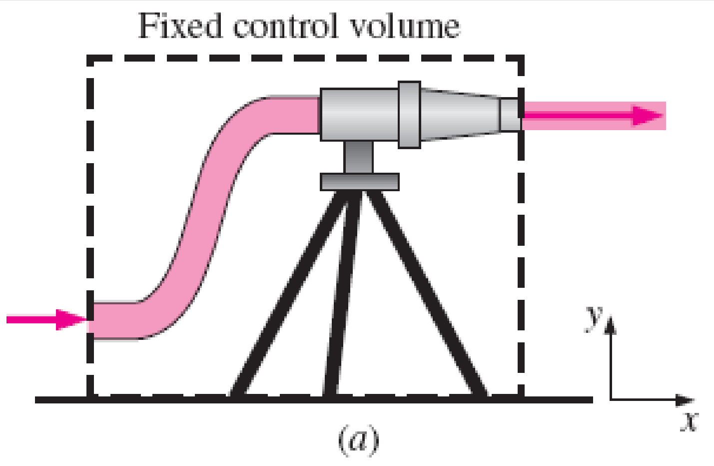


(c)

Many flow systems involve stationary hardware firmly fixed to a stationary surface, and such systems are best analyzed using ***fixed*** control volumes.

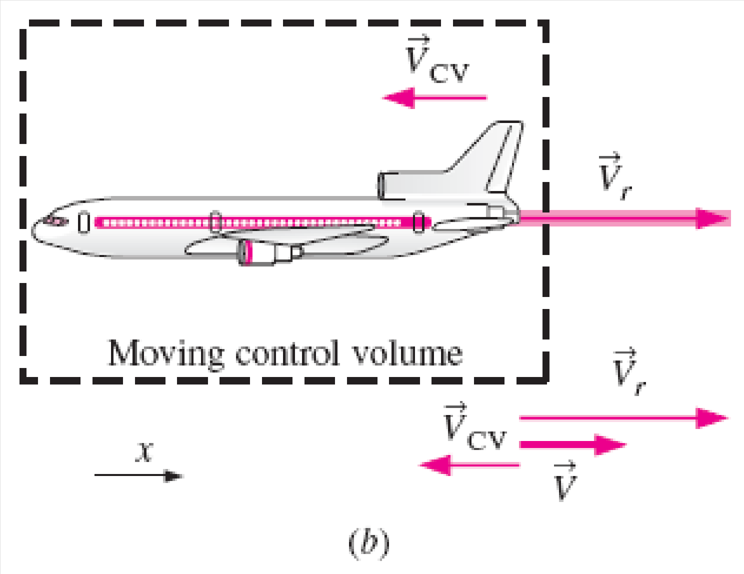
When analyzing flow systems that are moving or deforming, it is usually more convenient to allow the control volume to ***move*** or ***deform***.

In ***deforming*** control volume, part of the control surface moves relative to other parts.



(a)

Examples of (*a*) fixed, (*b*) moving, and (*c*) deforming control volumes.



(b)

6–3 ■ FORCES ACTING ON A CONTROL VOLUME

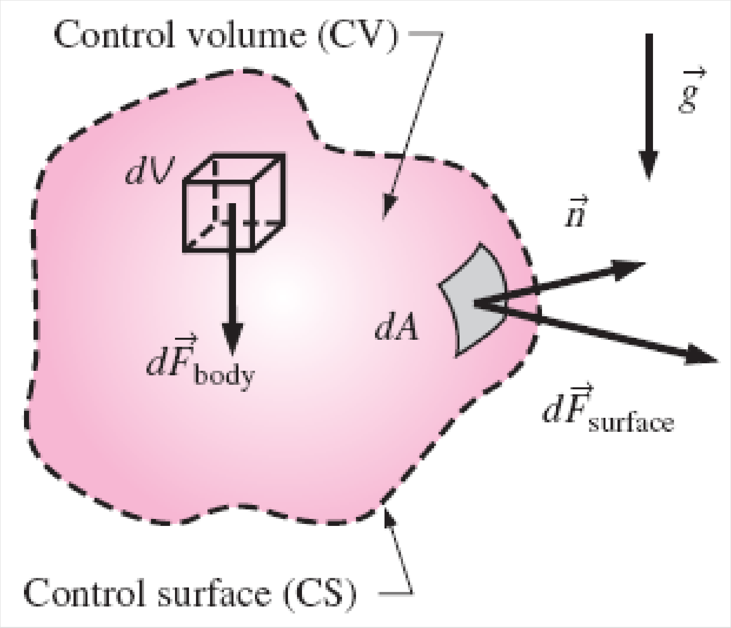
The forces acting on a control volume consist of

Body forces that act throughout the entire body of the control volume (such as gravity, electric, and magnetic forces) and

Surface forces that act on the control surface (such as pressure and viscous forces and reaction forces at points of contact).

Only external forces are considered in the analysis.

Total force acting on control volume: $\sum \vec{F} = \sum \vec{F}_{\text{body}} + \sum \vec{F}_{\text{surface}}$



The total force acting on a control volume is composed of body forces and surface forces; body force is shown on a differential volume element, and surface force is shown on a differential surface element.

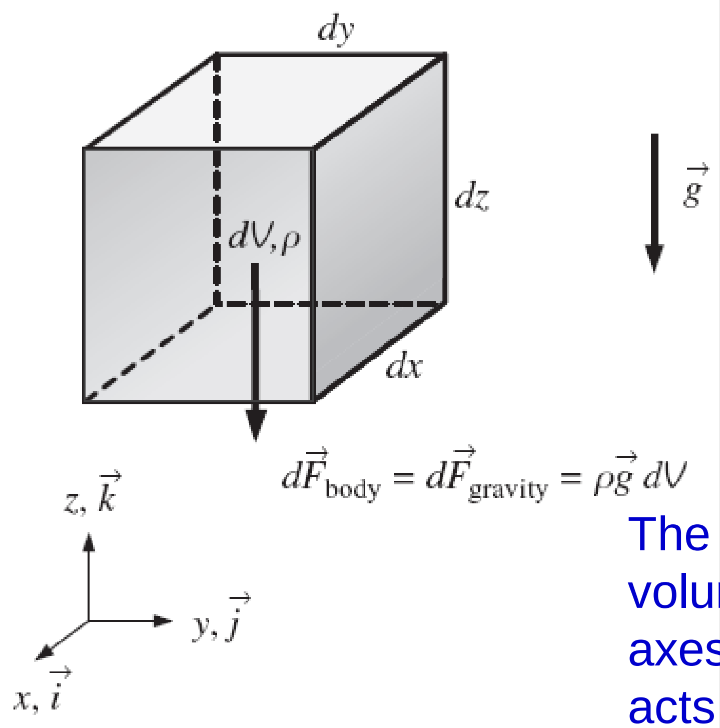
The most common body force is that of **gravity**, which exerts a downward force on every differential element of the control volume.

Gravitational force acting on a fluid element: $$d\vec{F}_{\text{gravity}} = \rho\vec{g}\, dV$$

Gravitational vector in Cartesian coordinates: $$\vec{g} = -g\vec{k}$$

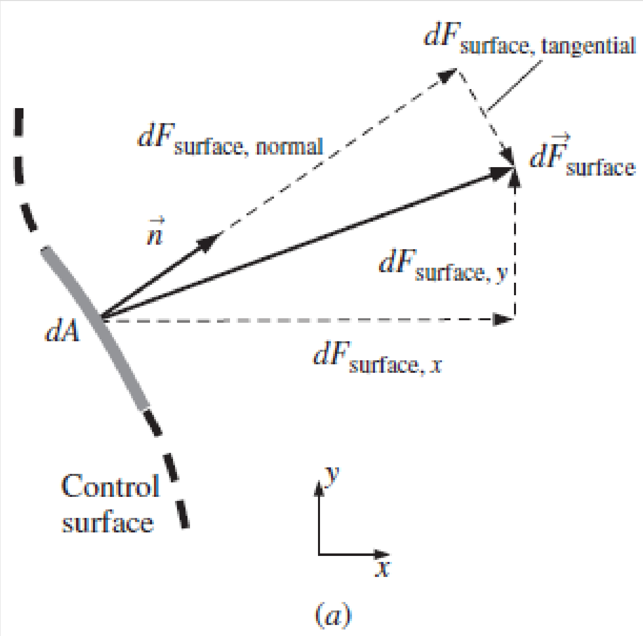
Total body force acting on control volume: $$\sum \vec{F}_{\text{body}} = \int_{\text{CV}} \rho\vec{g}\, dV = m_{\text{CV}}\vec{g}$$

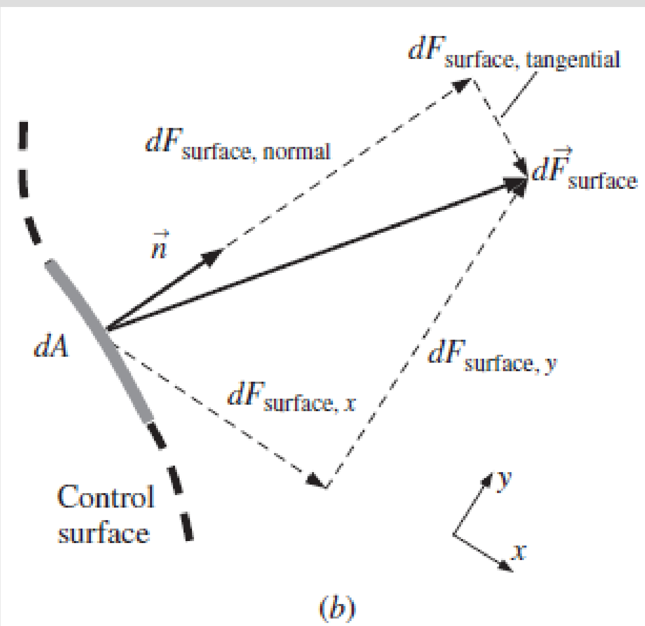
Total force: $$\underbrace{\sum \vec{F}}_{\text{total force}} = \underbrace{\sum \vec{F}_{\text{gravity}}}_{\text{body force}} + \underbrace{\sum \vec{F}_{\text{pressure}} + \sum \vec{F}_{\text{viscous}} + \sum \vec{F}_{\text{other}}}_{\text{surface forces}}$$



Surface forces are not as simple to analyze since they consist of both *normal* and *tangential* components.
Normal stresses are composed of pressure (which always acts inwardly normal) and viscous stresses.
Shear stresses are composed entirely of viscous stresses.

The gravitational force acting on a differential volume element of fluid is equal to its weight; the axes have been rotated so that the gravity vector acts *downward* in the negative *z*-direction.





Surface force acting on a differential surface element:

$$d\vec{F}_{\text{surface}} = \sigma_{ij} \cdot \vec{n}\, dA$$

Total surface force acting on control surface:

$$\sum \vec{F}_{\text{surface}} = \int_{CS} \sigma_{ij} \cdot \vec{n}\, dA$$

$$\sum \vec{F} = \sum \vec{F}_{\text{body}} + \sum \vec{F}_{\text{surface}} = \int_{CV} \rho \vec{g}\, dV + \int_{CS} \sigma_{ij} \cdot \vec{n}\, dA$$

Total force:

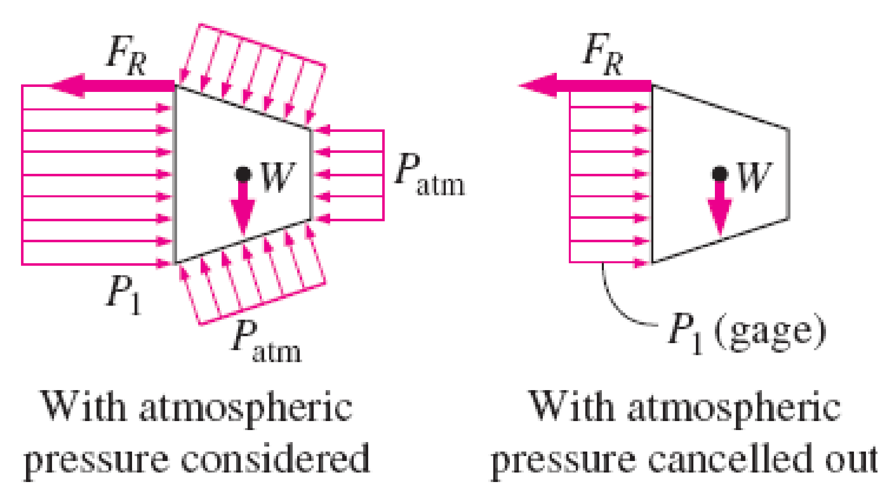
$$\underbrace{\sum \vec{F}}_{\text{total force}} = \underbrace{\sum \vec{F}_{\text{gravity}}}_{\text{body force}} + \underbrace{\sum \vec{F}_{\text{pressure}} + \sum \vec{F}_{\text{viscous}} + \sum \vec{F}_{\text{other}}}_{\text{surface forces}}$$

When coordinate axes are rotated (*a*) to (*b*), the components of the surface force change, even though the force itself remains the same; only two dimensions are shown here.

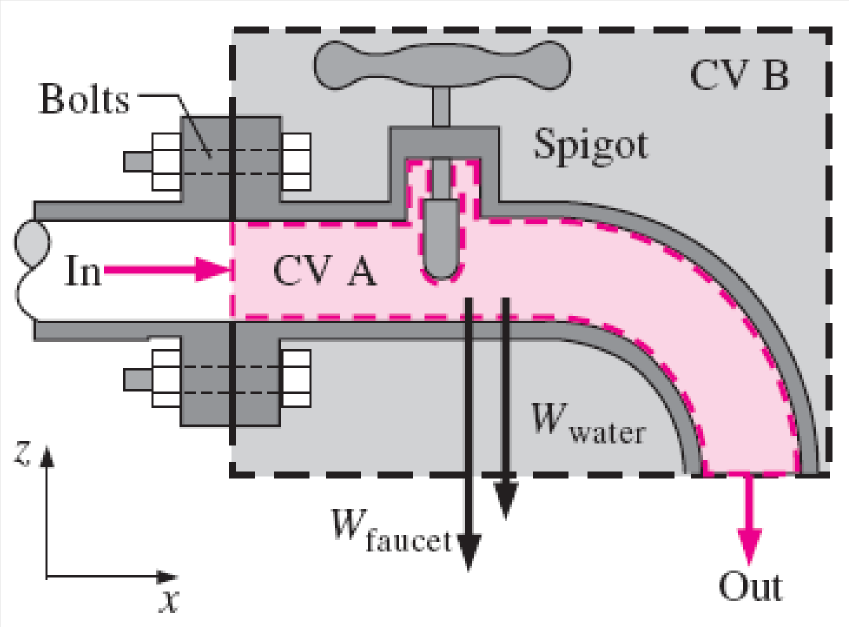
A common simplification in the application of Newton's laws of motion is to subtract the *atmospheric pressure* and work with gage pressures.

This is because atmospheric pressure acts in all directions, and its effect cancels out in every direction.

This means we can also ignore the pressure forces at outlet sections where the fluid is discharged to the atmosphere since the discharge pressure in such cases is very near atmospheric pressure at subsonic velocities.



Atmospheric pressure acts in all directions, and thus it can be ignored when performing force balances since its effect cancels out in every direction.



Cross section through a faucet assembly, illustrating the importance of choosing a control volume wisely; CV B is much easier to work with than CV A.

6–4 ■ THE LINEAR MOMENTUM EQUATION

Newton's second law for a system of mass m subjected to net force $\Sigma\vec{F}$ is expressed as

$$\sum \vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt} = \frac{d}{dt}(m\vec{V}) \qquad \textbf{(6–13)}$$

where $m\vec{V}$ is the **linear momentum** of the system. Noting that both the density and velocity may change from point to point within the system, Newton's second law can be expressed more generally as

$$\sum \vec{F} = \frac{d}{dt}\int_{\text{sys}} \rho\vec{V}\, dV \qquad \textbf{(6–14)}$$

where $\rho\vec{V}\, dV$ is the momentum of a differential element dV, which has mass $\delta m = \rho dV$.

Newton's second law can be stated as

The sum of all external forces acting on a system is equal to the time rate of change of linear momentum of the system.

This statement is valid for a coordinate system that is at rest or moves with a constant velocity, called an *inertial coordinate system* or *inertial reference frame*.

$$\frac{d(m\vec{V})_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, dV + \int_{CS} \rho \vec{V}\,(\vec{V}_r \cdot \vec{n})\, dA$$

$$\textit{General:} \quad \sum \vec{F} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, dV + \int_{CS} \rho \vec{V}(\vec{V}_r \cdot \vec{n})\, dA$$

$$\vec{V}_r = \vec{V} - \vec{V}_{CS}$$

$$\begin{pmatrix}\text{The sum of all}\\ \text{external forces}\\ \text{acting on a CV}\end{pmatrix} = \begin{pmatrix}\text{The time rate of change}\\ \text{of the linear momentum}\\ \text{of the contents of the CV}\end{pmatrix} + \begin{pmatrix}\text{The net flow rate of}\\ \text{linear momentum out of the}\\ \text{control surface by mass flow}\end{pmatrix}$$

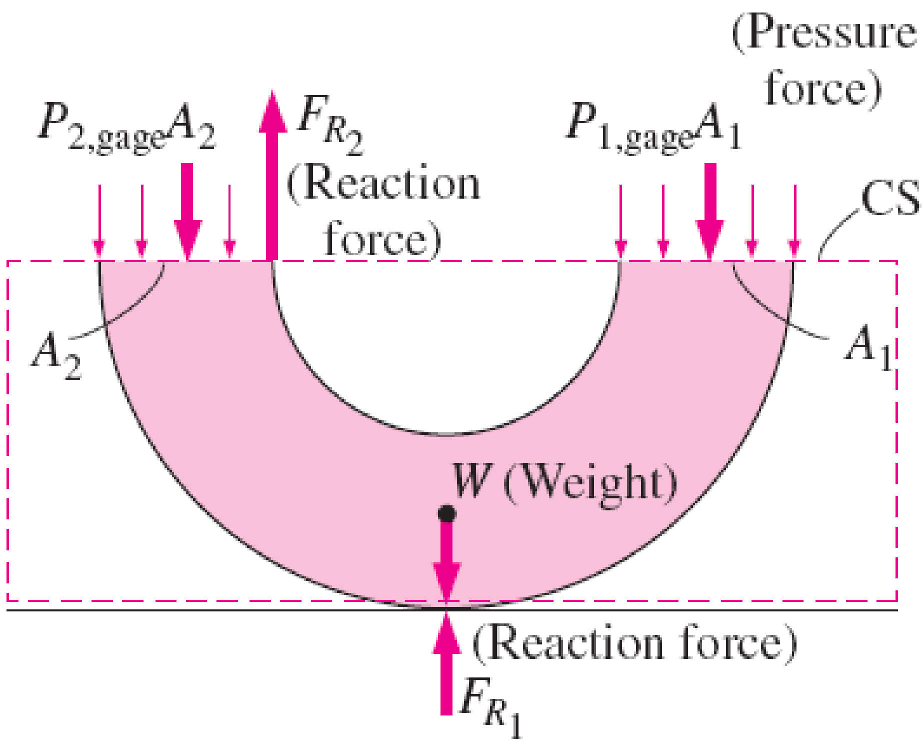
$$\textit{Fixed CV:} \quad \sum \vec{F} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, dV + \int_{CS} \rho \vec{V}(\vec{V} \cdot \vec{n})\, dA$$

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b\, dV + \int_{CS} \rho b(\vec{V}_r \cdot \vec{n})\, dA$$

$$B = m\vec{V} \qquad b = \vec{V} \qquad b = \vec{V}$$

$$\frac{d(m\vec{V})_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, dV + \int_{CS} \rho \vec{V}(\vec{V}_r \cdot \vec{n})\, dA$$

The linear momentum equation is obtained by replacing B in the Reynolds transport theorem by the momentum $m\vec{V}$, and b by the momentum per unit mass $\vec{V}$.



An 180° elbow supported by the ground

The momentum equation is commonly used to calculate the forces (usually on support systems or connectors) induced by the flow.

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, dV + \int_{CS} \rho \vec{V}(\vec{V}\cdot\vec{n})\, dA$$

In most flow systems, the sum of forces $\Sigma\vec{F}$ consists of weights, pressure forces, and reaction forces. Gage pressures are used here since atmospheric pressure cancels out on all sides of the control surface.

Special Cases

$$\sum \vec{F} = \int_{CS} \rho \vec{V} (\vec{V}_r \cdot \vec{n}) dA$$

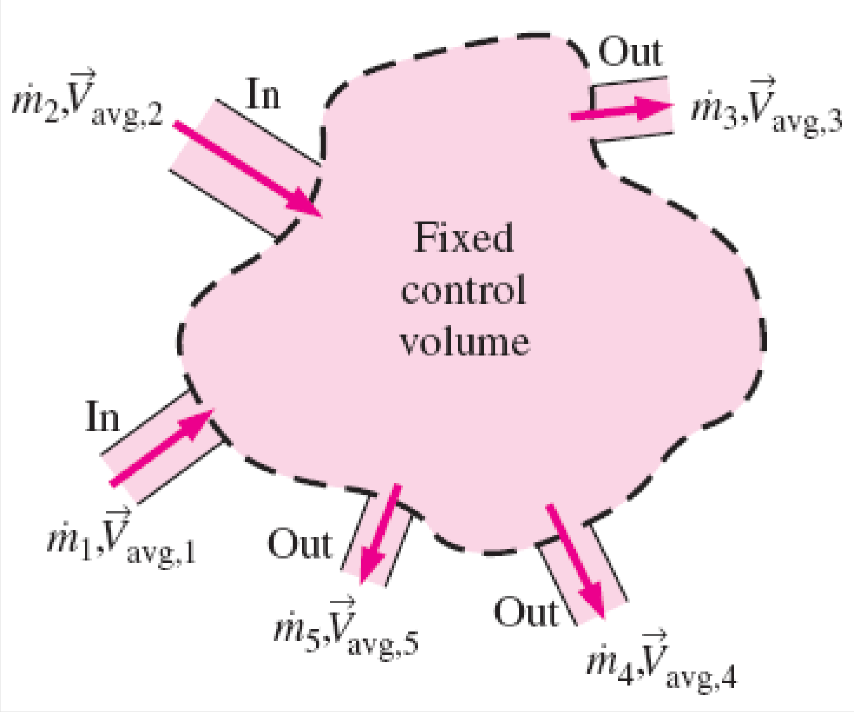
Steady flow

$$\dot{m} = \int_{A_c} \rho(\vec{V} \cdot \vec{n}) dA_c = \rho V_{avg} A_c$$

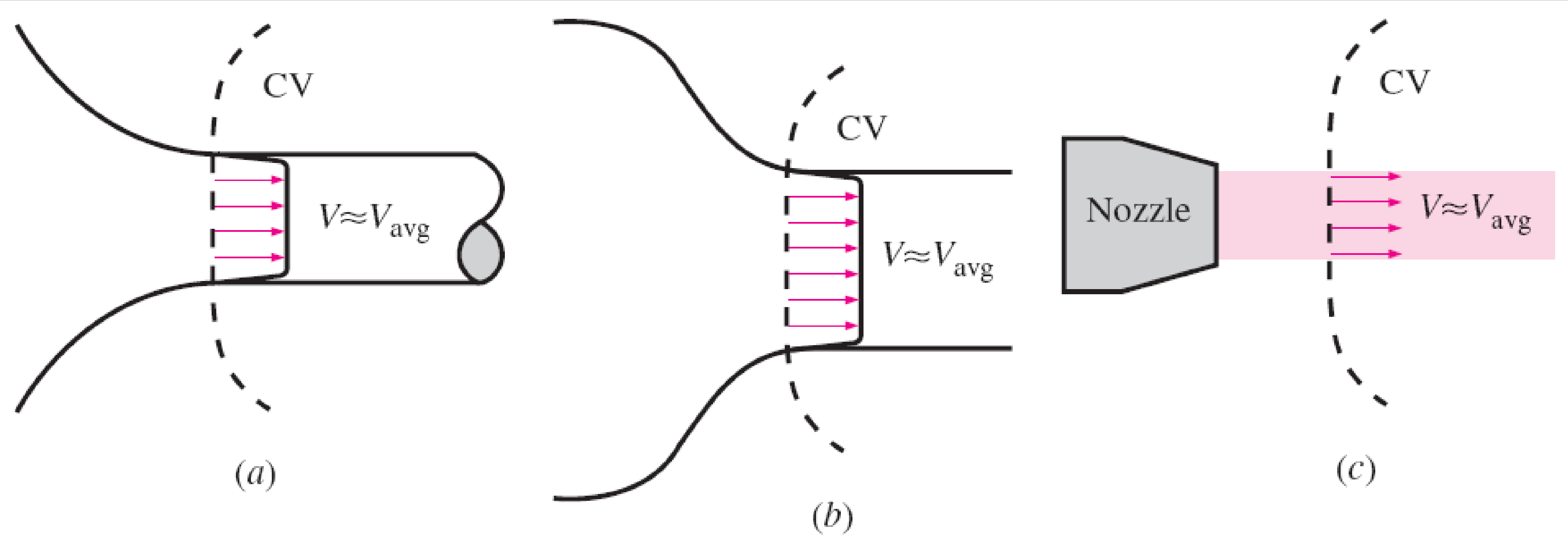
Mass flow rate across an inlet or outlet

$$\int_{A_c} \rho \vec{V}(\vec{V} \cdot \vec{n}) dA_c = \rho V_{avg} A_c \vec{V}_{avg} = \dot{m} \vec{V}_{avg}$$

Momentum flow rate across a uniform inlet or outlet:



In a typical engineering problem, the control volume may contain many inlets and outlets; at each inlet or outlet we define the mass flow rate and the average velocity.



Examples of inlets or outlets in which the uniform flow approximation is reasonable:
(*a*) the well-rounded entrance to a pipe,
(*b*) the entrance to a wind tunnel test section, and
(*c*) a slice through a free water jet in air.

Momentum-Flux Correction Factor, β

The velocity across most inlets and outlets is *not* uniform.

The control surface integral of Eq. 6–17 may be converted into algebraic form using a dimensionless correction factor β, called the **momentum-flux correction factor**.

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, dV + \int_{CS} \rho \vec{V}(\vec{V}\cdot\vec{n})\, dA \qquad (6\text{-}17)$$

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, dV + \sum_{\text{out}} \beta \dot{m} \vec{V}_{\text{avg}} - \sum_{\text{in}} \beta \dot{m} \vec{V}_{\text{avg}}$$

Momentum flux across an inlet or outlet: $\displaystyle \int_{A_c} \rho \vec{V}(\vec{V}\cdot\vec{n})\, dA_c = \beta \dot{m} \vec{V}_{\text{avg}}$

$$\beta = \frac{\int_{A_c} \rho V(\vec{V}\cdot\vec{n})\, dA_c}{\dot{m} V_{\text{avg}}} = \frac{\int_{A_c} \rho V(\vec{V}\cdot\vec{n})\, dA_c}{\rho V_{\text{avg}} A_c V_{\text{avg}}}$$

β is always greater than or equal to 1.

β is close to 1 for turbulent flow and not very close to 1 for fully developed laminar flow.

Momentum-flux correction factor: $\displaystyle \beta = \frac{1}{A_c}\int_{A_c} \left(\frac{V}{V_{\text{avg}}}\right)^2 dA_c$

EXAMPLE 6–1 **Momentum-Flux Correction Factor for Laminar Pipe Flow**

Consider laminar flow through a very long straight section of round pipe. It is shown in Chap. 8 that the velocity profile through a cross-sectional area of the pipe is parabolic (Fig. 6–15), with the axial velocity component given by

$$V = 2V_{avg}\left(1 - \frac{r^2}{R^2}\right) \qquad (1)$$

where R is the radius of the inner wall of the pipe and V_{avg} is the average velocity. Calculate the momentum-flux correction factor through a cross section of the pipe for the case in which the pipe flow represents an outlet of the control volume, as sketched in Fig. 6–15.

SOLUTION For a given velocity distribution we are to calculate the momentum-flux correction factor.

Assumptions **1** The flow is incompressible and steady. **2** The control volume slices through the pipe normal to the pipe axis, as sketched in Fig. 6–15.

Analysis We substitute the given velocity profile for V in Eq. 6–24 and integrate, noting that $dA_c = 2\pi r\,dr$,

$$\beta = \frac{1}{A_c}\int_{A_c}\left(\frac{V}{V_{avg}}\right)^2 dA_c = \frac{4}{\pi R^2}\int_0^R\left(1 - \frac{r^2}{R^2}\right)^2 2\pi r\,dr \qquad (2)$$

Defining a new integration variable $y = 1 - r^2/R^2$ and thus $dy = -2r\,dr/R^2$ (also, $y = 1$ at $r = 0$, and $y = 0$ at $r = R$) and performing the integration, the momentum-flux correction factor for fully developed laminar flow becomes

Laminar flow:
$$\beta = -4\int_1^0 y^2\,dy = -4\left[\frac{y^3}{3}\right]_1^0 = \frac{4}{3} \qquad (3)$$

Discussion We have calculated β for an outlet, but the same result would have been obtained if we had considered the cross section of the pipe as an *inlet* to the control volume.

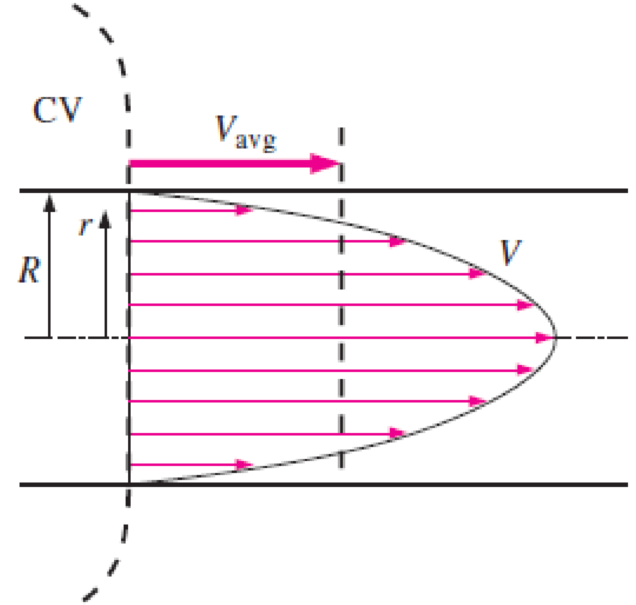


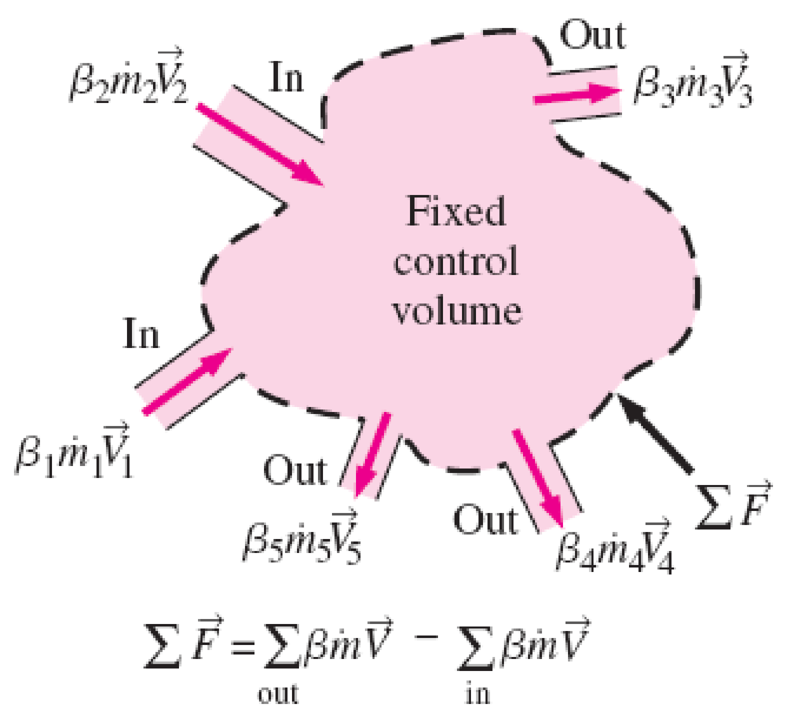
FIGURE 6–15
Velocity profile over a cross section of a pipe in which the flow is fully developed and laminar.

For turbulent flow β may have an insignificant effect at inlets and outlets, but for laminar flow β may be important and should not be neglected. It is wise to include β in all momentum control volume problems.

Steady Flow

Steady linear momentum equation: $\sum \vec{F} = \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$

The net force acting on the control volume during steady flow is equal to the difference between the rates of outgoing and incoming momentum flows.



The net force acting on the control volume during steady flow is equal to the difference between the outgoing and the incoming momentum fluxes.

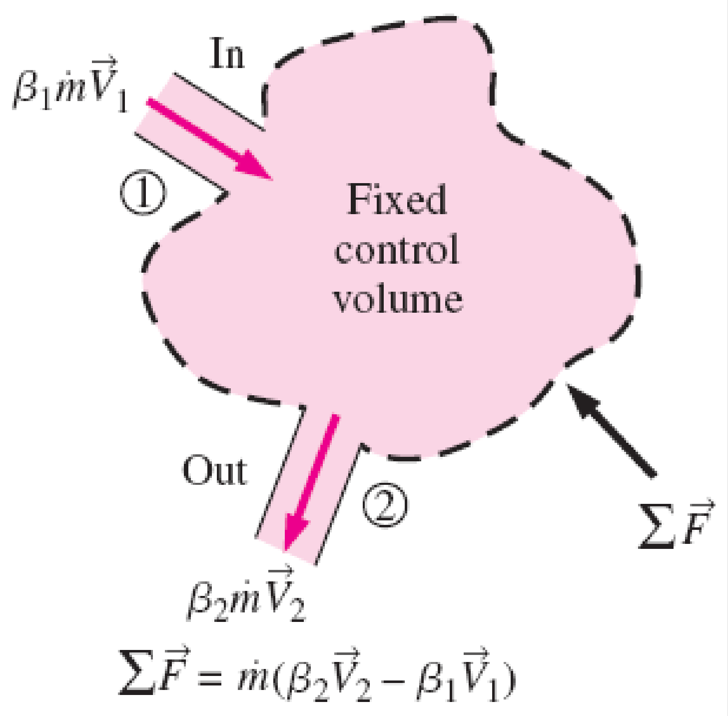
Steady Flow with One Inlet and One Outlet

$$\sum \vec{F} = \dot{m}\,(\beta_2 \vec{V}_2 - \beta_1 \vec{V}_1)$$

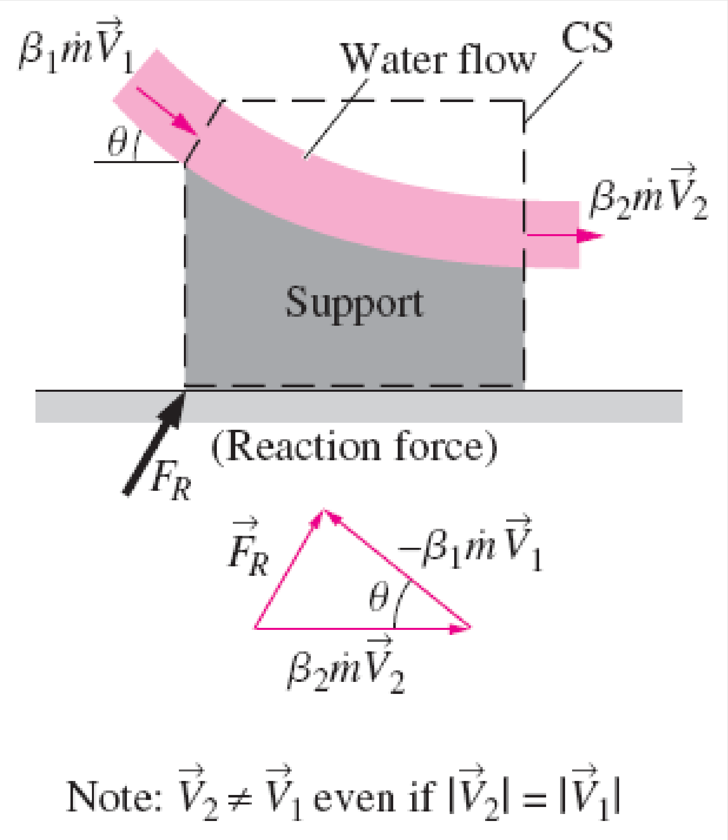
One inlet and one outlet

$$\sum F_x = \dot{m}(\beta_2 V_{2,x} - \beta_1 V_{1,x})$$

Along x-coordinate



A control volume with only one inlet and one outlet.



The determination by vector addition of the reaction force on the support caused by a change of direction of water.

Flow with No External Forces

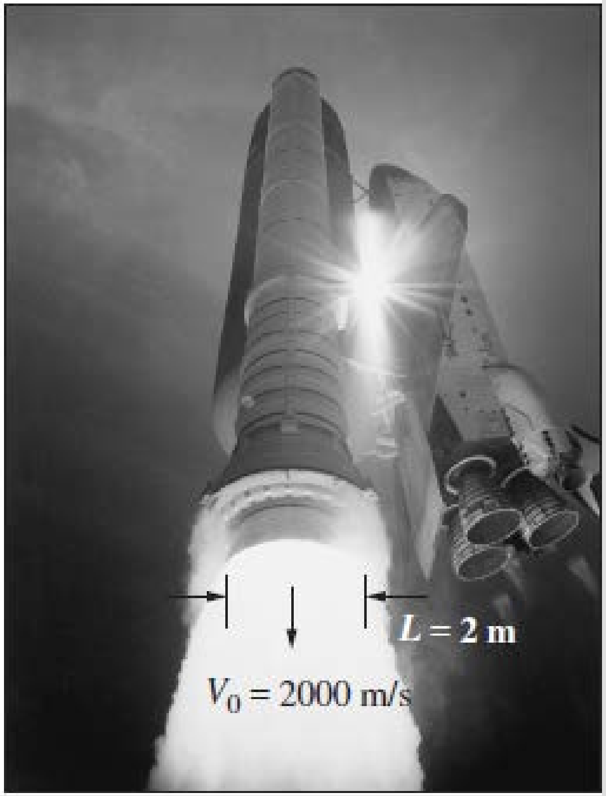
$$\text{No external forces:} \qquad 0 = \frac{d(m\vec{V})_{CV}}{dt} + \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$$

In the absence of external forces, the rate of change of the momentum of a control volume is equal to the difference between the rates of incoming and outgoing momentum flow rates.

$$\frac{d(m\vec{V})_{CV}}{dt} = m_{CV}\frac{d\vec{V}_{CV}}{dt} = (m\vec{a})_{CV} = m_{CV}\vec{a}$$

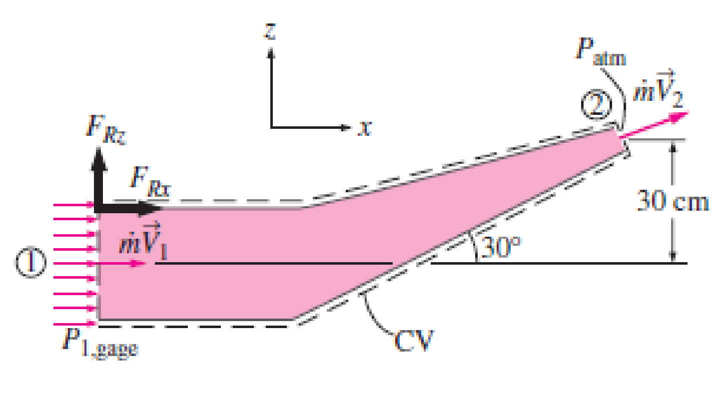
$$\vec{F}_{\text{thrust}} = m_{CV}\vec{a} = \sum_{\text{in}} \beta \dot{m} \vec{V} - \sum_{\text{out}} \beta \dot{m} \vec{V}$$

The thrust needed to lift the space shuttle is generated by the rocket engines as a result of momentum change of the fuel as it is accelerated from about zero to an exit speed of about 2000 m/s after combustion.



EXAMPLE 6–2 The Force to Hold a Deflector Elbow in Place

A reducing elbow is used to deflect water flow at a rate of 14 kg/s in a horizontal pipe upward 30° while accelerating it (Fig. 6–20). The elbow discharges water into the atmosphere. The cross-sectional area of the elbow is 113 cm² at the inlet and 7 cm² at the outlet. The elevation difference between the centers of the outlet and the inlet is 30 cm. The weight of the elbow and the water in it is considered to be negligible. Determine (*a*) the gage pressure at the center of the inlet of the elbow and (*b*) the anchoring force needed to hold the elbow in place.



SOLUTION A reducing elbow deflects water upward and discharges it to the atmosphere. The pressure at the inlet of the elbow and the force needed to hold the elbow in place are to be determined.

Assumptions **1** The flow is steady, and the frictional effects are negligible. **2** The weight of the elbow and the water in it is negligible. **3** The water is discharged to the atmosphere, and thus the gage pressure at the outlet is zero. **4** The flow is turbulent and fully developed at both the inlet and outlet of the control volume, and we take the momentum-flux correction factor to be $\beta = 1.03$ (as a conservative estimate) at both the inlet and the outlet.

Properties We take the density of water to be 1000 kg/m³.

Analysis (*a*) We take the elbow as the control volume and designate the inlet by 1 and the outlet by 2. We also take the *x*- and *z*-coordinates as shown. The continuity equation for this one-inlet, one-outlet, steady-flow system is $\dot{m}_1 = \dot{m}_2 = \dot{m} = 14$ kg/s. Noting that $\dot{m} = \rho AV$, the inlet and outlet velocities of water are

$$V_1 = \frac{\dot{m}}{\rho A_1} = \frac{14\ \text{kg/s}}{(1000\ \text{kg/m}^3)(0.0113\ \text{m}^2)} = 1.24\ \text{m/s}$$

$$V_2 = \frac{\dot{m}}{\rho A_2} = \frac{14\ \text{kg/s}}{(1000\ \text{kg/m}^3)(7 \times 10^{-4}\ \text{m}^2)} = 20.0\ \text{m/s}$$

We use the Bernoulli equation (Chap. 5) as a first approximation to calculate the pressure. In Chap. 8 we will learn how to account for frictional losses along the walls. Taking the center of the inlet cross section as the reference level ($z_1 = 0$) and noting that $P_2 = P_{atm}$, the Bernoulli equation for a streamline going through the center of the elbow is expressed as

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

$$P_1 - P_2 = \rho g\left(\frac{V_2^2 - V_1^2}{2g} + z_2 - z_1\right)$$

$$P_1 - P_{atm} = (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2) \times \left(\frac{(20\ \text{m/s})^2 - (1.24\ \text{m/s})^2}{2(9.81\ \text{m/s}^2)} + 0.3 - 0\right)\left(\frac{1\ \text{kN}}{1000\ \text{kg} \cdot \text{m/s}^2}\right)$$

$$P_{1,\ \text{gage}} = 202.2\ \text{kN/m}^2 = \mathbf{202.2\ kPa} \quad \text{(gage)}$$

(b) The momentum equation for steady flow is

$$\sum \vec{F} = \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$$

We let the x- and z-components of the anchoring force of the elbow be F_{Rx} and F_{Rz}, and assume them to be in the positive direction. We also use gage pressure since the atmospheric pressure acts on the entire control surface. Then the momentum equations along the x- and z-axes become

$$F_{Rx} + P_{1,\,\text{gage}} A_1 = \beta \dot{m} V_2 \cos\theta - \beta \dot{m} V_1$$
$$F_{Rz} = \beta \dot{m} V_2 \sin\theta$$

where we have set $\beta = \beta_1 = \beta_2$. Solving for F_{Rx} and F_{Rz}, and substituting the given values,

$$\begin{aligned} F_{Rx} &= \beta \dot{m}(V_2 \cos\theta - V_1) - P_{1,\,\text{gage}} A_1 \\ &= 1.03(14 \text{ kg/s})[(20 \cos 30^\circ - 1.24) \text{ m/s}]\left(\frac{1 \text{ N}}{1 \text{ kg} \cdot \text{m/s}^2}\right) \\ &\quad - (202{,}200 \text{ N/m}^2)(0.0113 \text{ m}^2) \\ &= 232 - 2285 = \mathbf{-2053 \text{ N}} \end{aligned}$$

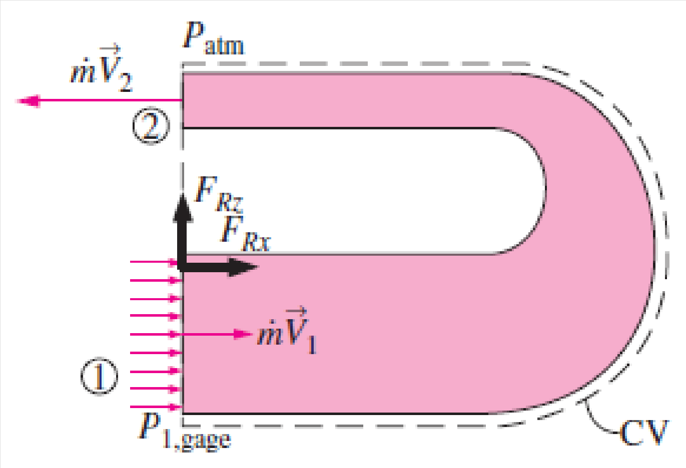
$$F_{Rz} = \beta \dot{m} V_2 \sin\theta = (1.03)(14 \text{ kg/s})(20 \sin 30^\circ \text{ m/s})\left(\frac{1 \text{ N}}{1 \text{ kg} \cdot \text{m/s}^2}\right) = \mathbf{144 \text{ N}}$$

The negative result for F_{Rx} indicates that the assumed direction is wrong, and it should be reversed. Therefore, F_{Rx} acts in the negative x-direction.

Discussion There is a nonzero pressure distribution along the inside walls of the elbow, but since the control volume is outside the elbow, these pressures do not appear in our analysis. The weight of the elbow and the water in it could be added to the vertical force for better accuracy. The actual value of $P_{1,\,\text{gage}}$ will be higher than that calculated here because of frictional and other irreversible losses in the elbow.

EXAMPLE 6–3 The Force to Hold a Reversing Elbow in Place

The deflector elbow in Example 6–2 is replaced by a reversing elbow such that the fluid makes a 180° U-turn before it is discharged, as shown in Fig. 6–21. The elevation difference between the centers of the inlet and the exit sections is still 0.3 m. Determine the anchoring force needed to hold the elbow in place.



SOLUTION The inlet and the outlet velocities and the pressure at the inlet of the elbow remain the same, but the vertical component of the anchoring force at the connection of the elbow to the pipe is zero in this case ($F_{Rz} = 0$) since there is no other force or momentum flux in the vertical direction (we are neglecting the weight of the elbow and the water). The horizontal component of the anchoring force is determined from the momentum equation written in the x-direction. Noting that the outlet velocity is negative since it is in the negative x-direction, we have

$$F_{Rx} + P_{1,\,gage}A_1 = \beta_2\dot{m}(-V_2) - \beta_1\dot{m}V_1 = -\beta\dot{m}(V_2 + V_1)$$

Solving for F_{Rx} and substituting the known values,

$$F_{Rx} = -\beta\dot{m}(V_2 + V_1) - P_{1,\,gage}A_1$$

$$= -(1.03)(14\ \text{kg/s})[(20 + 1.24)\ \text{m/s}]\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right) - (202{,}200\ \text{N/m}^2)(0.0113\ \text{m}^2)$$

$$= -306 - 2285 = \mathbf{-2591\ N}$$

Therefore, the horizontal force on the flange is 2591 N acting in the negative x-direction (the elbow is trying to separate from the pipe). This force is equivalent to the weight of about 260 kg mass, and thus the connectors (such as bolts) used must be strong enough to withstand this force.

Discussion The reaction force in the x-direction is larger than that of Example 6–2 since the walls turn the water over a much greater angle. If the reversing elbow is replaced by a straight nozzle (like one used by firefighters) such that water is discharged in the positive x-direction, the momentum equation in the x-direction becomes

$$F_{Rx} + P_{1,\,gage}A_1 = \beta\dot{m}V_2 - \beta\dot{m}V_1 \quad\rightarrow\quad F_{Rx} = \beta\dot{m}(V_2 - V_1) - P_{1,\,gage}A_1$$

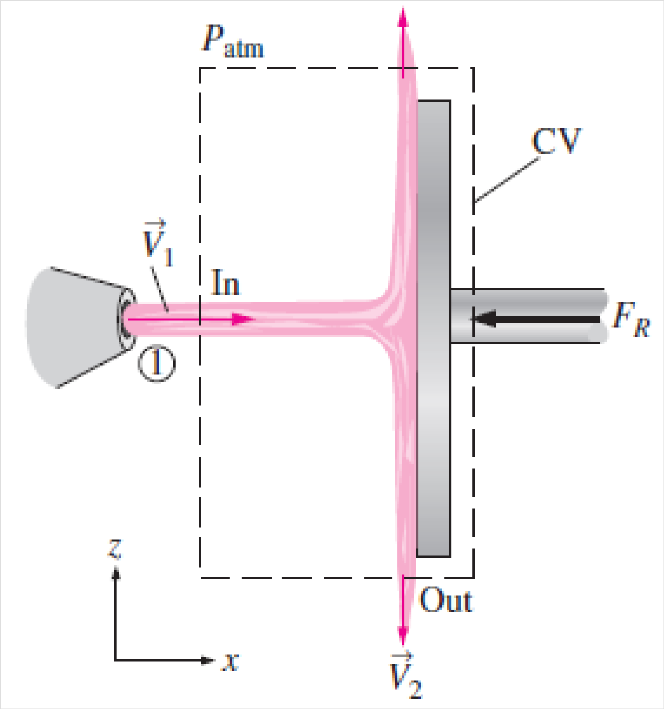
since both V_1 and V_2 are in the positive x-direction. This shows the importance of using the correct sign (positive if in the positive direction and negative if in the opposite direction) for velocities and forces.

EXAMPLE 6–4 **Water Jet Striking a Stationary Plate**

Water is accelerated by a nozzle to an average speed of 20 m/s, and strikes a stationary vertical plate at a rate of 10 kg/s with a normal velocity of 20 m/s (Fig. 6–22). After the strike, the water stream splatters off in all directions in the plane of the plate. Determine the force needed to prevent the plate from moving horizontally due to the water stream.

SOLUTION A water jet strikes a vertical stationary plate normally. The force needed to hold the plate in place is to be determined.

Assumptions 1 The flow of water at the nozzle outlet is steady. **2** The water splatters in directions normal to the approach direction of the water jet.



3 The water jet is exposed to the atmosphere, and thus the pressure of the water jet and the splattered water leaving the control volume is atmospheric pressure, which is disregarded since it acts on the entire system. **4** The vertical forces and momentum fluxes are not considered since they have no effect on the horizontal reaction force. **5** The effect of the momentum-flux correction factor is negligible, and thus $\beta \cong 1$ at the inlet.

Analysis We draw the control volume for this problem such that it contains the entire plate and cuts through the water jet and the support bar normally. The momentum equation for steady one-dimensional flow is given as

$$\sum \vec{F} = \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V} \tag{1}$$

Writing Eq. 1 for this problem along the x-direction (without forgetting the negative sign for forces and velocities in the negative x-direction) and noting that $V_{1,x} = V_1$ and $V_{2,x} = 0$ gives

$$-F_R = 0 - \beta \dot{m} V_1$$

Substituting the given values,

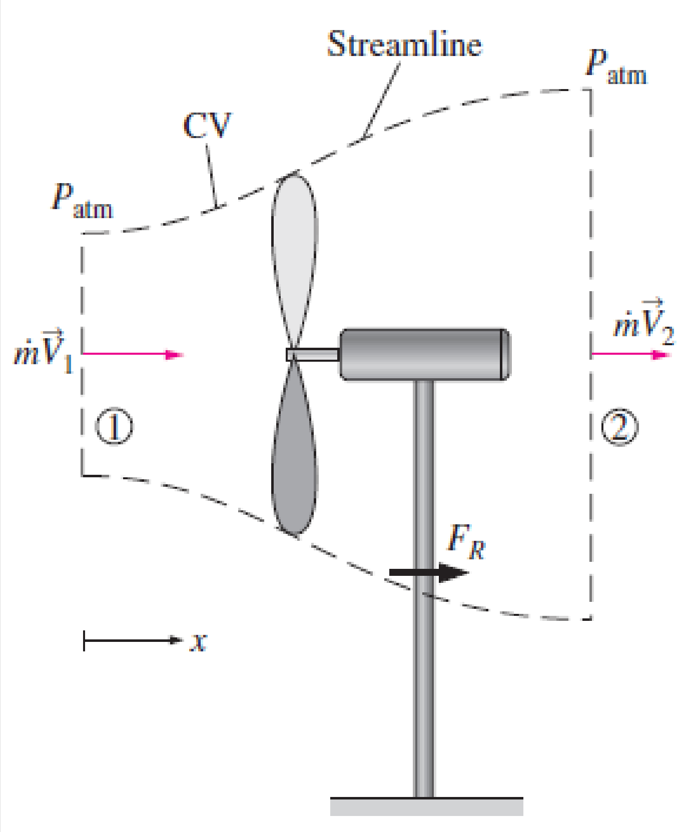
$$F_R = \beta \dot{m} V_1 = (1)(10 \text{ kg/s})(20 \text{ m/s})\left(\frac{1 \text{ N}}{1 \text{ kg} \cdot \text{m/s}^2}\right) = 200 \text{ N}$$

Therefore, the support must apply a 200-N horizontal force (equivalent to the weight of about a 20-kg mass) in the negative x-direction (the opposite direction of the water jet) to hold the plate in place.

Discussion The plate absorbs the full brunt of the momentum of the water jet since the x-direction momentum at the outlet of the control volume is zero. If the control volume were drawn instead along the interface between the water and the plate, there would be additional (unknown) pressure forces in the analysis. By cutting the control volume through the support, we avoid having to deal with this additional complexity. This is an example of a "wise" choice of control volume.

EXAMPLE 6–5 Power Generation and Wind Loading of a Wind Turbine

A wind generator with a 30-ft-diameter blade span has a cut-in wind speed (minimum speed for power generation) of 7 mph, at which velocity the turbine generates 0.4 kW of electric power (Fig. 6–23). Determine (*a*) the efficiency of the wind turbine–generator unit and (*b*) the horizontal force exerted by the wind on the supporting mast of the wind turbine. What is the effect of doubling the wind velocity to 14 mph on power generation and the force exerted? Assume the efficiency remains the same, and take the density of air to be 0.076 lbm/ft³.



Analysis Kinetic energy is a mechanical form of energy, and thus it can be converted to work entirely. Therefore, the power potential of the wind is proportional to its kinetic energy, which is $V^2/2$ per unit mass, and thus the maximum power is $\dot{m}V^2/2$ for a given mass flow rate:

$$V_1 = (7 \text{ mph})\left(\frac{1.4667 \text{ ft/s}}{1 \text{ mph}}\right) = 10.27 \text{ ft/s}$$

$$\dot{m} = \rho_1 V_1 A_1 = \rho_1 V_1 \frac{\pi D^2}{4} = (0.076 \text{ lbm/ft}^3)(10.27 \text{ ft/s})\frac{\pi(30 \text{ ft})^2}{4} = 551.7 \text{ lbm/s}$$

$$\dot{W}_{max} = \dot{m}\text{ke}_1 = \dot{m}\frac{V_1^2}{2}$$

$$= (551.7 \text{ lbm/s})\frac{(10.27 \text{ ft/s})^2}{2}\left(\frac{1 \text{ lbf}}{32.2 \text{ lbm}\cdot\text{ft/s}^2}\right)\left(\frac{1 \text{ kW}}{737.56 \text{ lbf}\cdot\text{ft/s}}\right)$$

$$= 1.225 \text{ kW}$$

Therefore, the available power to the wind turbine is 1.225 kW at the wind velocity of 7 mph. Then the turbine–generator efficiency becomes

$$\eta_{\text{wind turbine}} = \frac{\dot{W}_{act}}{\dot{W}_{max}} = \frac{0.4 \text{ kW}}{1.225 \text{ kW}} = 0.327 \qquad (\text{or } 32.7\%)$$

(*b*) The frictional effects are assumed to be negligible, and thus the portion of incoming kinetic energy not converted to electric power leaves the wind turbine as outgoing kinetic energy. Noting that the mass flow rate remains constant, the exit velocity is determined to be

$$\dot{m}\text{ke}_2 = \dot{m}\text{ke}_1(1 - \eta_{\text{wind turbine}}) \quad \rightarrow \quad \dot{m}\frac{V_2^2}{2} = \dot{m}\frac{V_1^2}{2}(1 - \eta_{\text{wind turbine}}) \qquad \textbf{(1)}$$

or

$$V_2 = V_1\sqrt{1 - \eta_{\text{wind turbine}}} = (10.27 \text{ ft/s})\sqrt{1 - 0.327} = 8.43 \text{ ft/s}$$

We draw a control volume around the wind turbine such that the wind is normal to the control surface at the inlet and the outlet and the entire control surface is at atmospheric pressure (Fig. 6–23). The momentum equation for steady one-dimensional flow is given as

$$\sum \vec{F} = \sum_{\text{out}} \beta\dot{m}\vec{V} - \sum_{\text{in}} \beta\dot{m}\vec{V} \qquad \textbf{(2)}$$

Writing Eq. 1 along the *x*-direction and noting that $\beta = 1$, $V_{1,x} = V_1$, and $V_{2,x} = V_2$ give

$$F_R = \dot{m}V_2 - \dot{m}V_1 = \dot{m}(V_2 - V_1) \qquad \textbf{(3)}$$

Substituting the known values into Eq. 3 gives

$$F_R = \dot{m}(V_2 - V_1) = (551.7 \text{ lbm/s})(8.43 - 10.27 \text{ ft/s})\left(\frac{1 \text{ lbf}}{32.2 \text{ lbm} \cdot \text{ft/s}^2}\right)$$

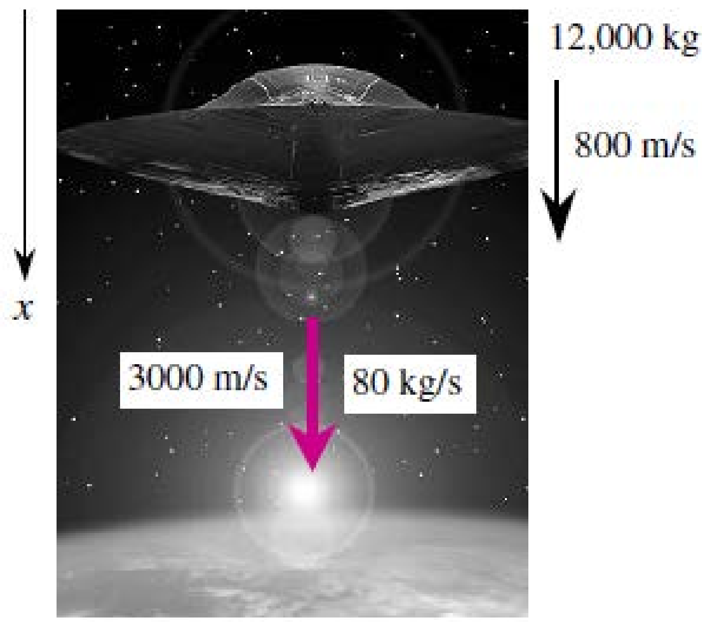
$$= -31.5 \text{ lbf}$$

The negative sign indicates that the reaction force acts in the negative *x*-direction, as expected. Then the force exerted by the wind on the mast becomes $F_{\text{mast}} = -F_R =$ **31.5 lbf.**

The power generated is proportional to V^3 since the mass flow rate is proportional to V and the kinetic energy to V^2. Therefore, doubling the wind velocity to 14 mph will increase the power generation by a factor of $2^3 = 8$ to $0.4 \times 8 = 3.2$ kW. The force exerted by the wind on the support mast is proportional to V^2. Therefore, doubling the wind velocity to 14 mph will increase the wind force by a factor of $2^2 = 4$ to $31.5 \times 4 = 126$ lbf.

EXAMPLE 6–6 **Deceleration of a Spacecraft**

A spacecraft with a mass of 12,000 kg is dropping vertically towards a planet at a constant speed of 800 m/s (Fig. 6–24). To slow down the spacecraft, a solid-fuel rocket at the bottom is fired, and combustion gases leave the rocket at a constant rate of 80 kg/s and at a velocity of 3000 m/s relative to the spacecraft in the direction of motion of the spacecraft for a period of 5 s. Disregarding the small change in the mass of the spacecraft, determine (*a*) the deceleration of the spacecraft during this period, (*b*) the change of velocity of the spacecraft, and (*c*) the thrust exerted on the spacecraft.



Analysis (*a*) For convenience, we choose an inertial reference frame that moves with the spacecraft at the same initial velocity. Then the velocities of fluid stream relative to an inertial reference frame become simply the velocities relative to the spacecraft. We take the direction of motion of the spacecraft as the positive direction along the *x*-axis. There are no external forces acting on the spacecraft, and its mass is essentially constant. Therefore, the spacecraft can be treated as a solid body with constant mass, and the momentum equation in this case is, from Eq. 6–29,

$$\vec{F}_{\text{thrust}} = m_{\text{spacecraft}} \vec{a}_{\text{spacecraft}} = \sum_{\text{in}} \beta \dot{m} \vec{V} - \sum_{\text{out}} \beta \dot{m} \vec{V}$$

where the fluid stream velocities relative to the inertial reference frame in this case are identical to the velocities relative to the spacecraft. Noting that the motion is on a straight line and the discharged gases move in the positive *x*-direction, we write the momentum equation using magnitudes as

$$m_{\text{spacecraft}} a_{\text{spacecraft}} = m_{\text{spacecraft}} \frac{dV_{\text{spacecraft}}}{dt} = -\dot{m}_{\text{gas}} V_{\text{gas}}$$

Noting that gases leave in the positive *x* direction and substituting, the acceleration of the spacecraft during the first 5 seconds is determined to be

$$a_{\text{spacecraft}} = \frac{dV_{\text{spacecraft}}}{dt} = -\frac{\dot{m}_{\text{gas}}}{m_{\text{spacecraft}}}V_{\text{gas}} = -\frac{80 \text{ kg/s}}{12{,}000 \text{ kg}}(+3000 \text{ m/s}) = \mathbf{-20 \text{ m/s}^2}$$

The negative value confirms that the spacecraft is decelerating in the positive x direction at a rate of 20 m/s^2.

(*b*) Knowing the deceleration, which is constant, the velocity change of the spacecraft during the first 5 seconds is determined from the definition of acceleration to be

$$dV_{\text{spacecraft}} = a_{\text{spacecraft}}dt \rightarrow \Delta V_{\text{spacecraft}} = a_{\text{spacecraft}}\Delta t = (-20 \text{ m/s}^2)(5 \text{ s})$$
$$= \mathbf{-100 \text{ m/s}}$$

(*c*) The thrusting force exerted on the space aircraft is, from Eq. 6-29,

$$F_{\text{thrust}} = 0 - \dot{m}_{\text{gas}}V_{\text{gas}} = 0 - (80 \text{ kg/s})(+3000 \text{ m/s})\left(\frac{1 \text{ kN}}{1000 \text{ kg} \cdot \text{m/s}^2}\right) = \mathbf{-240 \text{ kN}}$$

The negative sign indicates that the trusting force due to firing of the rocket acts on the aircraft in the negative x-direction.

Discussion Note that if this fired rocket were attached somewhere on a test stand, it would exert a force of 240 kN (equivalent to the weight of about 24 tons of mass) to its support in the opposite direction of the discharged gases.

EXAMPLE 6–7 Net Force on a Flange

Water flows at a rate of 18.5 gal/min through a flanged faucet with a partially closed gate valve spigot (Fig. 6–25). The inner diameter of the pipe at the location of the flange is 0.780 in (= 0.0650 ft), and the pressure at that location is measured to be 13.0 psig. The total weight of the faucet assembly plus the water within it is 12.8 lbf. Calculate the net force on the flange.

Analysis We choose the faucet and its immediate surroundings as the control volume, as shown in Fig. 6–25 along with all the forces acting on it. These forces include the weight of the water and the weight of the faucet assembly, the gage pressure force at the inlet to the control volume, and the net force of the flange on the control volume, which we call $\vec{F}_R$. We use gage pressure for convenience since the gage pressure on the rest of the control surface is zero (atmospheric pressure). Note that the pressure through the outlet of the control volume is also atmospheric since we are assuming incompressible flow; hence, the gage pressure is also zero through the outlet.

We now apply the control volume conservation laws. Conservation of mass is trivial here since there is only one inlet and one outlet; namely, the mass flow rate into the control volume is equal to the mass flow rate out of the control volume. Also, the outflow and inflow average velocities are identical since the inner diameter is constant and the water is incompressible, and are determined to be

$$V_2 = V_1 = V = \frac{\dot{V}}{A_c} = \frac{\dot{V}}{\pi D^2/4} = \frac{18.5\ \text{gal/min}}{\pi(0.065\ \text{ft})^2/4}\left(\frac{0.1337\ \text{ft}^3}{1\ \text{gal}}\right)\left(\frac{1\ \text{min}}{60\ \text{s}}\right) = 12.42\ \text{ft/s}$$

Also,

$$\dot{m} = \rho\dot{V} = (62.3\ \text{lbm/ft}^3)(18.5\ \text{gal/min})\left(\frac{0.1337\ \text{ft}^3}{1\ \text{gal}}\right)\left(\frac{1\ \text{min}}{60\ \text{s}}\right) = 2.568\ \text{lbm/s}$$

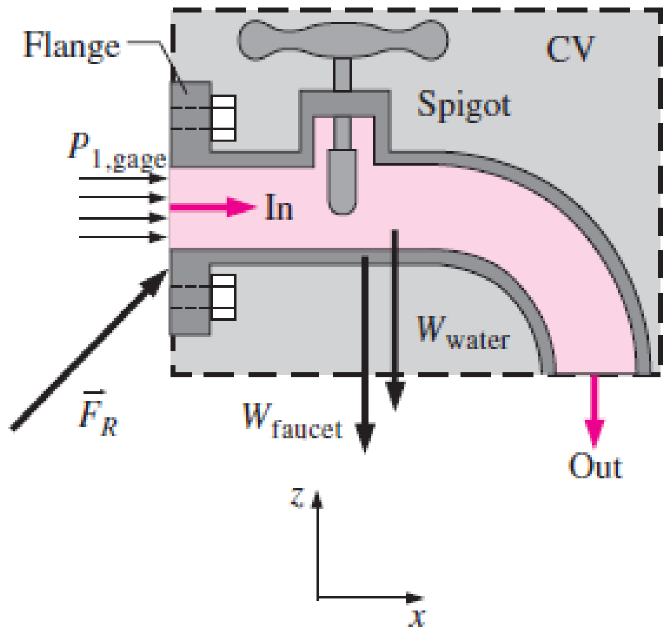


FIGURE 6–25
Control volume for Example 6–7 with all forces shown; gage pressure is used for convenience.

Next we apply the momentum equation for steady flow,

$$\sum \vec{F} = \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V} \tag{1}$$

We let the x- and z-components of the force acting on the flange be F_{Rx} and F_{Rz}, and assume them to be in the positive directions. The magnitude of the velocity in the x-direction is $+V_1$ at the inlet, but zero at the outlet. The magnitude of the velocity in the z-direction is zero at the inlet, but $-V_2$ at the outlet. Also, the weight of the faucet assembly and the water within it acts in the $-z$-direction as a body force. No pressure or viscous forces act on the chosen (wise) control volume in the z-direction.

The components of Eq. 1 along the x- and z-directions become

$$F_{Rx} + P_{1,\text{ gage}} A_1 = 0 - \dot{m}(+V_1)$$

$$F_{Rz} - W_{\text{faucet}} - W_{\text{water}} = \dot{m}(-V_2) - 0$$

Solving for F_{Rx} and F_{Rz}, and substituting the given values,

$$\begin{aligned} F_{Rx} &= -\dot{m}V_1 - P_{1,\text{ gage}} A_1 \\ &= -(2.568 \text{ lbm/s})(12.42 \text{ ft/s})\left(\frac{1 \text{ lbf}}{32.2 \text{ lbm} \cdot \text{ft/s}^2}\right) - (13 \text{ lbf/in}^2)\frac{\pi(0.780 \text{ in})^2}{4} \\ &= -7.20 \text{ lbf} \end{aligned}$$

$$\begin{aligned} F_{Rz} &= -\dot{m}V_2 + W_{\text{faucet+water}} \\ &= -(2.568 \text{ lbm/s})(12.42 \text{ ft/s})\left(\frac{1 \text{ lbf}}{32.2 \text{ lbm} \cdot \text{ft/s}^2}\right) + 12.8 \text{ lbf} = 11.8 \text{ lbf} \end{aligned}$$

Then the net force of the flange on the control volume is expressed in vector form as

$$\vec{F}_R = F_{Rx}\vec{i} + F_{Rz}\vec{k} = -7.20\vec{i} + 11.8\vec{k} \quad \text{lbf}$$

From Newton's third law, the force the faucet assembly exerts on the flange is the negative of $\vec{F}_R$,

$$\vec{F}_{\text{faucet on flange}} = -\vec{F}_R = \mathbf{7.20\vec{i} - 11.8\vec{k} \quad lbf}$$

6–5 ■ REVIEW OF ROTATIONAL MOTION AND ANGULAR MOMENTUM

Rotational motion: A motion during which all points in the body move in circles about the axis of rotation.

Rotational motion is described with angular quantities such as the angular distance θ, angular velocity ω, and angular acceleration α.

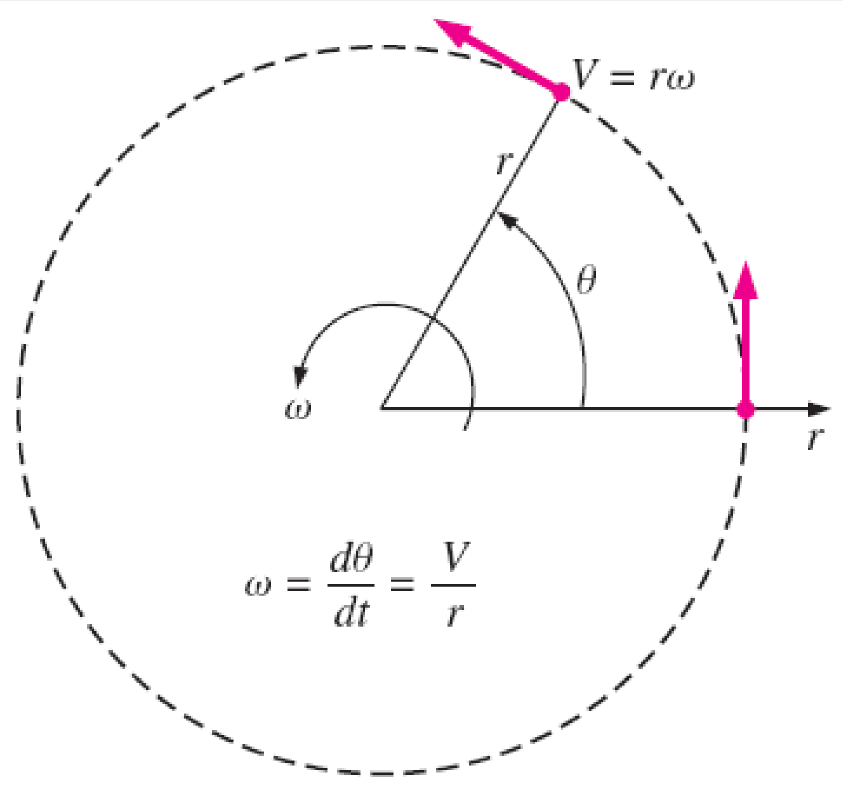
Angular velocity: The angular distance traveled per unit time.

Angular acceleration: The rate of change of angular velocity.

$$\omega = \frac{d\theta}{dt} = \frac{d(l/r)}{dt} = \frac{1}{r}\frac{dl}{dt} = \frac{V}{r}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \frac{1}{r}\frac{dV}{dt} = \frac{a_t}{r}$$

$V = r\omega$ and $a_t = r\alpha$



The relations between angular distance θ, angular velocity ω, and linear velocity V.

- Newton's second law requires that there must be a force acting in the tangential direction to cause angular acceleration.
- The strength of the rotating effect, called the *moment* or *torque*, is proportional to the magnitude of the force and its distance from the axis of rotation.
- The perpendicular distance from the axis of rotation to the line of action of the force is called the *moment arm*, and the torque *M* acting on a point mass *m* at a normal distance *r* from the axis of rotation is expressed as

$$M = rF_t = rma_t = mr^2\alpha$$

Torque

$$M = \int_{\text{mass}} r^2\alpha\,\delta m = \left[\int_{\text{mass}} r^2\,\delta m\right]\alpha = I\alpha$$

I is the *moment of inertia* of the body about the axis of rotation, which is a measure of the inertia of a body against rotation.

Unlike mass, the rotational inertia of a body also depends on the distribution of the mass of the body with respect to the axis of rotation.

Mass, m	$\longleftrightarrow$	Moment of inertia, I
Linear acceleration, $\vec{a}$	$\longleftrightarrow$	Angular acceleration, $\vec{\alpha}$
Linear velocity, $\vec{V}$	$\longleftrightarrow$	Angular velocity, $\vec{\omega}$
Linear momentum	$\longleftrightarrow$	Angular momentum
$m\vec{V}$	$\longleftrightarrow$	$I\vec{\omega}$
Force, $\vec{F}$	$\longleftrightarrow$	Torque, $\vec{M}$
$\vec{F} = m\vec{a}$	$\longleftrightarrow$	$\vec{M} = I\vec{\alpha}$
Moment of force, $\vec{M}$	$\longleftrightarrow$	Moment of momentum, $\vec{H}$
$\vec{M} = \vec{r} \times \vec{F}$	$\longleftrightarrow$	$\vec{H} = \vec{r} \times m\vec{V}$

Analogy between corresponding linear and angular quantities.

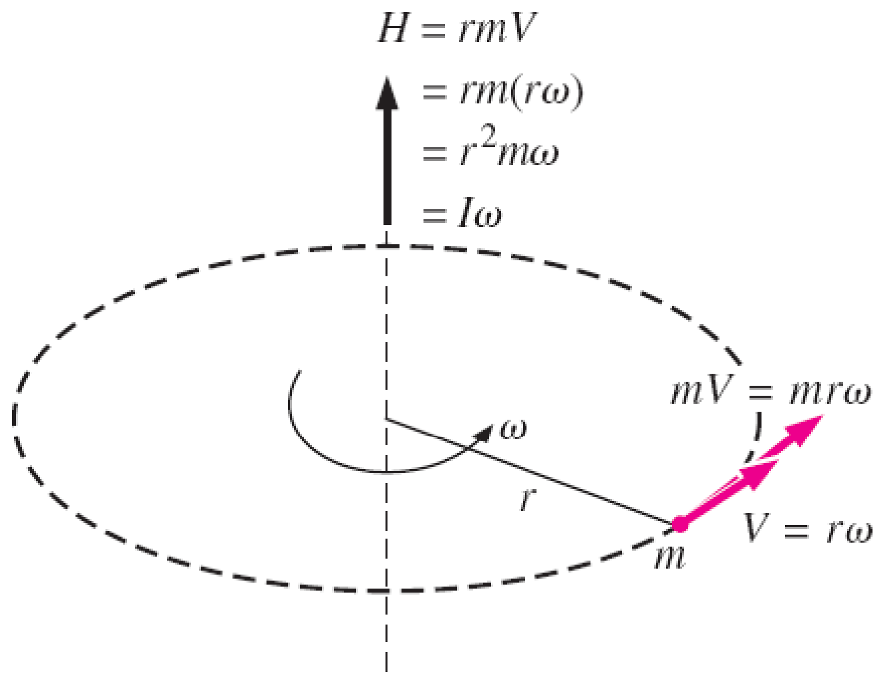
$$H = \int_{\text{mass}} r^2\omega\,\delta m = \left[\int_{\text{mass}} r^2\,\delta m\right]\omega = I\omega$$

Angular momentum

$$\vec{H} = I\vec{\omega}$$

$$\vec{M} = I\vec{\alpha} = I\frac{d\vec{\omega}}{dt} = \frac{d(I\vec{\omega})}{dt} = \frac{d\vec{H}}{dt}$$

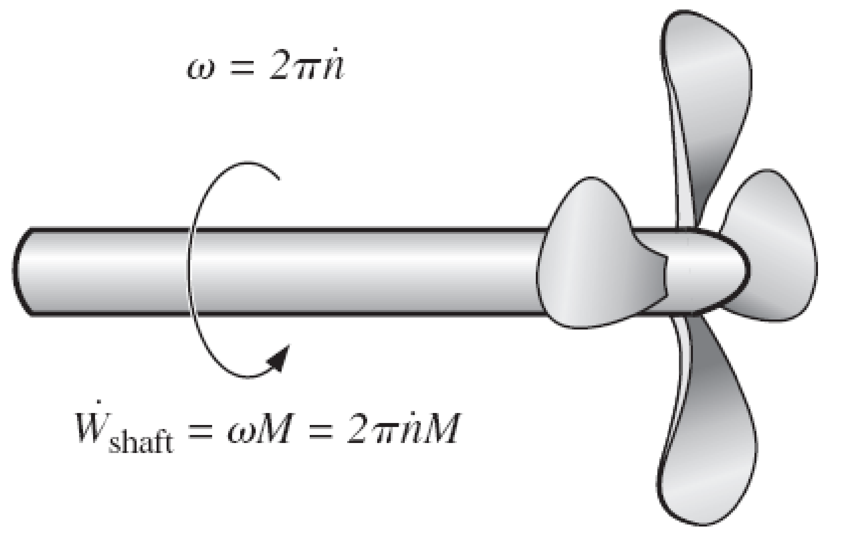
Angular momentum equation



Angular momentum of point mass *m* rotating at angular velocity ω at distance *r* from the axis of rotation.

$$\omega = \frac{2\pi\dot{n}}{60} \qquad \text{(rad/s)}$$

Angular velocity versus rpm



The relations between angular velocity, rpm, and the power transmitted through a shaft.

$$\dot{W}_{\text{shaft}} = FV = Fr\omega = M\omega$$

$$\dot{W}_{\text{shaft}} = \omega M = 2\pi \dot{n} M \qquad \text{(W)}$$ Shaft power

$$\text{KE}_r = \tfrac{1}{2} I\omega^2$$ Rotational kinetic energy

During rotational motion, the direction of velocity changes even when its magnitude remains constant. Velocity is a vector quantity, and thus a change in direction constitutes a change in velocity with time, and thus acceleration. This is called **centripetal acceleration**.

$$a_r = \frac{V^2}{r} = r\omega^2$$

Centripetal acceleration is directed toward the axis of rotation (opposite direction of radial acceleration), and thus the radial acceleration is negative. Centripetal acceleration is the result of a force acting on an element of the body toward the axis of rotation, known as the **centripetal force**, whose magnitude is $F_r = mV^2/r$.

Tangential and radial accelerations are perpendicular to each other, and the total linear acceleration is determined by their vector sum:

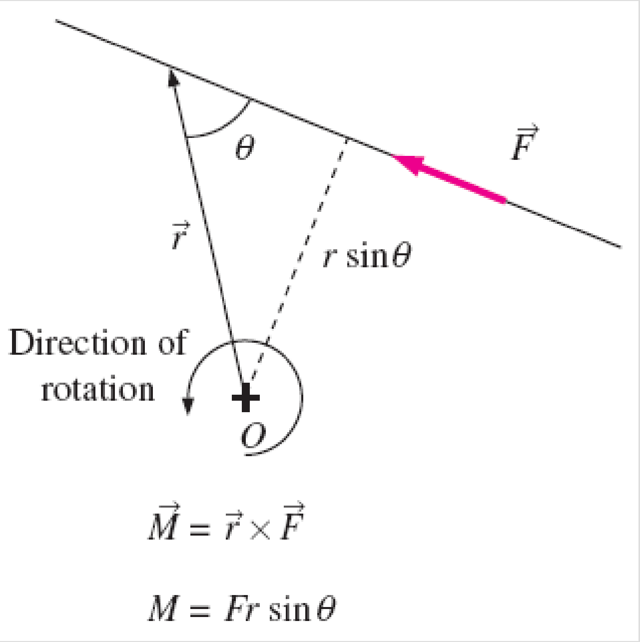
$$\vec{a} = \vec{a}_t + \vec{a}_r$$

6–6 ■ THE ANGULAR MOMENTUM EQUATION

Many engineering problems involve the moment of the linear momentum of flow streams, and the rotational effects caused by them.

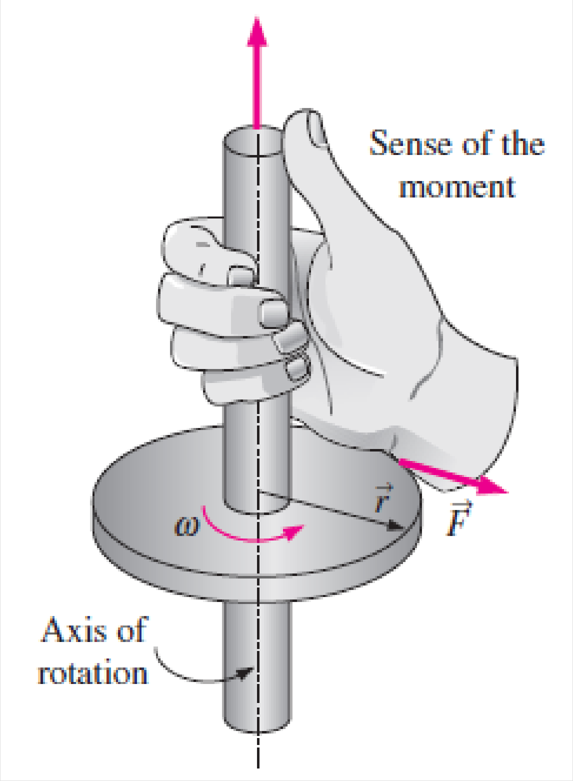
Such problems are best analyzed by the *angular momentum equation*, also called the *moment of momentum equation.*

An important class of fluid devices, called *turbomachines*, which include centrifugal pumps, turbines, and fans, is analyzed by the angular momentum equation.



The moment of a force $\vec{F}$ about a point O is the vector product of the position vector $\vec{r}$ and $\vec{F}$.

A force whose line of action passes through point *O* produces zero moment about point *O*.



The determination of the direction of the moment by the right-hand rule.

Moment of momentum

$$\vec{H} = \vec{r} \times m\vec{V}$$

Moment of momentum (system)

$$\vec{H}_{\text{sys}} = \int_{\text{sys}} (\vec{r} \times \vec{V})\rho\, dV$$

$$\frac{d\vec{H}_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{sys}} (\vec{r} \times \vec{V})\rho\, dV$$

Rate of change of moment of momentum

$$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho b\, dV + \int_{\text{CS}} \rho b(\vec{V}_r \cdot \vec{n})\, dA$$

$$B = \vec{H} \qquad b = \vec{r} \times \vec{V} \qquad b = \vec{r} \times \vec{V}$$

$$\frac{d\vec{H}_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} (\vec{r} \times \vec{V})\rho\, dV + \int_{\text{CS}} (\vec{r} \times \vec{V})\rho(\vec{V}_r \cdot \vec{n})\, dA$$

The angular momentum equation is obtained by replacing B in the Reynolds transport theorem by the angular momentum $\vec{H}$, and b by the angular momentum per unit mass $\vec{r} \times \vec{V}$.

Angular momentum equation for a system

$$\sum \vec{M} = \frac{d\vec{H}_{\text{sys}}}{dt}$$

$$\sum \vec{M} = \sum (\vec{r} \times \vec{F})$$

$$\frac{d\vec{H}_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} (\vec{r} \times \vec{V})\rho\, dV + \int_{\text{CS}} (\vec{r} \times \vec{V})\rho(\vec{V}_r \cdot \vec{n})\, dA$$

General: $$\sum \vec{M} = \frac{d}{dt}\int_{\text{CV}} (\vec{r} \times \vec{V})\rho\, dV + \int_{\text{CS}} (\vec{r} \times \vec{V})\rho(\vec{V}_r \cdot \vec{n})\, dA$$

$$\begin{pmatrix}\text{The sum of all}\\ \text{external moments}\\ \text{acting on a CV}\end{pmatrix} = \begin{pmatrix}\text{The time rate of change}\\ \text{of the angular momentum}\\ \text{of the contents of the CV}\end{pmatrix} + \begin{pmatrix}\text{The net flow rate of}\\ \text{angular momentum}\\ \text{out of the control}\\ \text{surface by mass flow}\end{pmatrix}$$

Fixed CV: $$\sum \vec{M} = \frac{d}{dt}\int_{\text{CV}} (\vec{r} \times \vec{V})\rho\, dV + \int_{\text{CS}} (\vec{r} \times \vec{V})\rho(\vec{V} \cdot \vec{n})\, dA$$

Special Cases

During *steady flow*, the amount of angular momentum within the control volume remains constant, and thus the time rate of change of angular momentum of the contents of the control volume is zero.

Steady flow: $$\sum \vec{M} = \int_{CS} (\vec{r} \times \vec{V})\rho(\vec{V}_r \cdot \vec{n})\,dA$$

An approximate form of the angular momentum equation in terms of average properties at inlets and outlets:

$$\sum \vec{M} = \frac{d}{dt}\int_{CV} (\vec{r} \times \vec{V})\rho\,dV + \sum_{\text{out}} \vec{r} \times \dot{m}\vec{V} - \sum_{\text{in}} \vec{r} \times \dot{m}\vec{V}$$

Steady flow: $$\sum \vec{M} = \sum_{\text{out}} \vec{r} \times \dot{m}\vec{V} - \sum_{\text{in}} \vec{r} \times \dot{m}\vec{V}$$

The net torque acting on the control volume during steady flow is equal to the difference between the outgoing and incoming angular momentum flow rates.

$$\sum M = \sum_{\text{out}} r\dot{m}V - \sum_{\text{in}} r\dot{m}V$$ *scalar form of angular momentum equation*

Flow with No External Moments

No external moments:
$$0 = \frac{d\vec{H}_{\mathrm{CV}}}{dt} + \sum_{\mathrm{out}} (\vec{r} \times \dot{m}\vec{V}) - \sum_{\mathrm{in}} (\vec{r} \times \dot{m}\vec{V})$$

In the absence of external moments, the rate of change of the angular momentum of a control volume is equal to the difference between the incoming and outgoing angular momentum fluxes.

When the moment of inertia *I* of the control volume remains constant, the irst term on the right side of the above equation becomes simply moment of inertia times angular acceleration. Therefore, the control volume in this case can be treated as a solid body, with a net torque of

$$\vec{M}_{\mathrm{body}} = I_{\mathrm{body}}\vec{\alpha} = \sum_{\mathrm{in}} (\vec{r} \times \dot{m}\vec{V}) - \sum_{\mathrm{out}} (\vec{r} \times \dot{m}\vec{V})$$

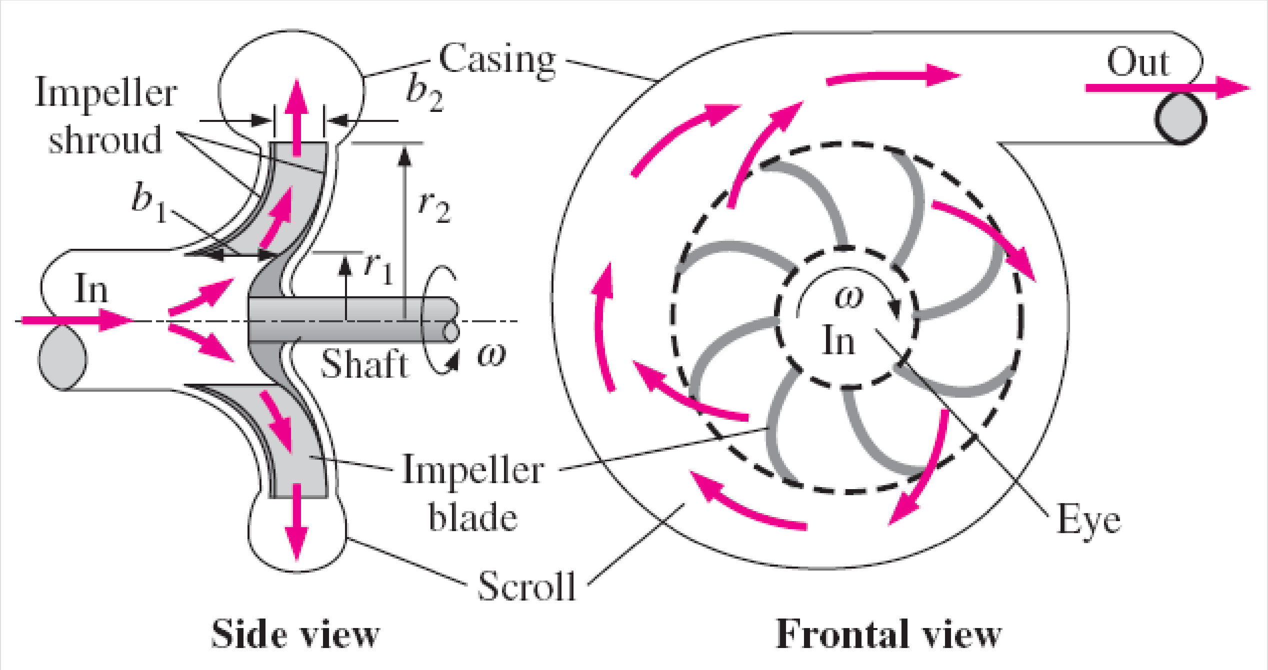
This approach can be used to determine the angular acceleration of space vehicles and aircraft when a rocket is fired in a direction different than the direction of motion.

Radial-Flow Devices

Radial-flow devices: Many rotary-flow devices such as centrifugal pumps and fans involve flow in the radial direction normal to the axis of rotation.

Axial-flow devices are easily analyzed using the linear momentum equation.

Radial-flow devices involve large changes in angular momentum of the fluid and are best analyzed with the help of the angular momentum equation.



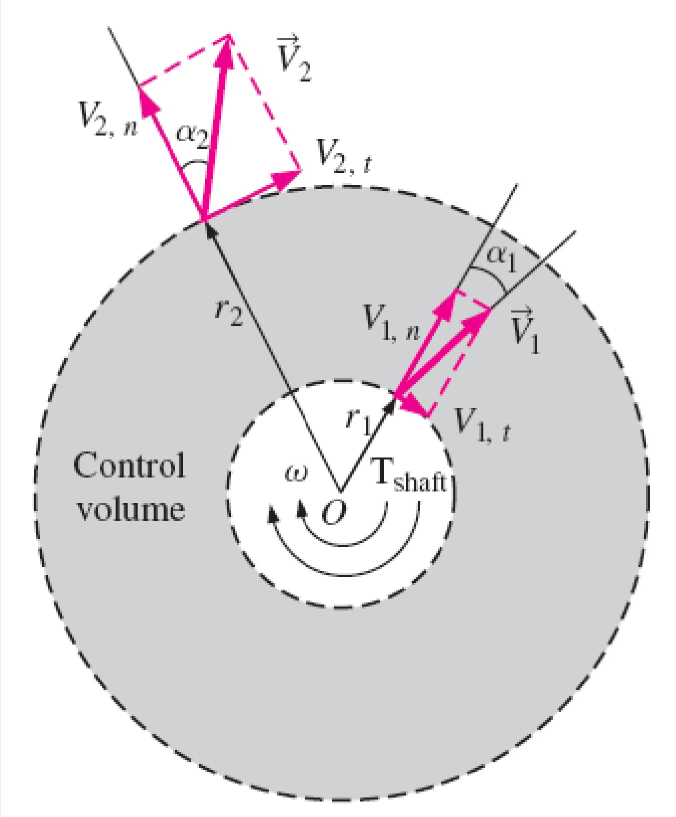
Side and frontal views of a typical centrifugal pump.

The conservation of mass equation for steady incompressible flow

$$\dot{V}_1 = \dot{V}_2 = \dot{V} \quad \rightarrow \quad (2\pi r_1 b_1)V_{1,n} = (2\pi r_2 b_2)V_{2,n}$$

$$V_{1,n} = \frac{\dot{V}}{2\pi r_1 b_1} \quad \text{and} \quad V_{2,n} = \frac{\dot{V}}{2\pi r_2 b_2}$$

$$\sum M = \sum_{\text{out}} r\dot{m}V - \sum_{\text{in}} r\dot{m}V$$ angular momentum equation



$$T_{\text{shaft}} = \dot{m}(r_2 V_{2,t} - r_1 V_{1,t})$$ Euler's turbine formula

$$T_{\text{shaft}} = \dot{m}(r_2 V_2 \sin\alpha_2 - r_1 V_1 \sin\alpha_1)$$

When $V_{1,t} = \omega r_1$ $V_{2,t} = \omega r_2$

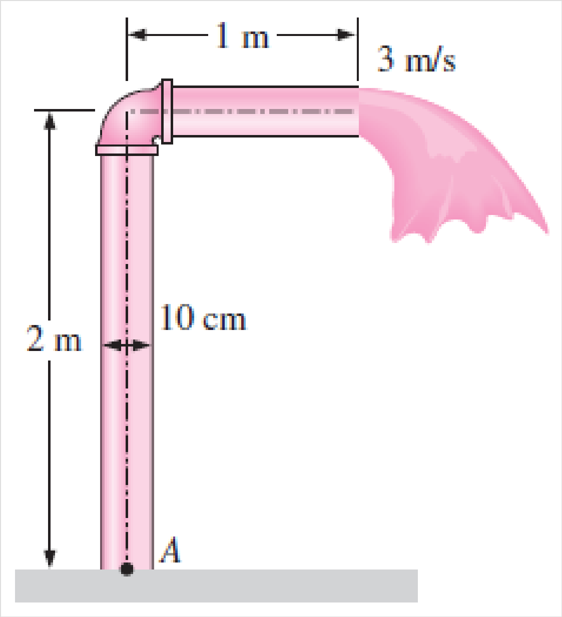
$$T_{\text{shaft, ideal}} = \dot{m}\omega(r_2^2 - r_1^2)$$

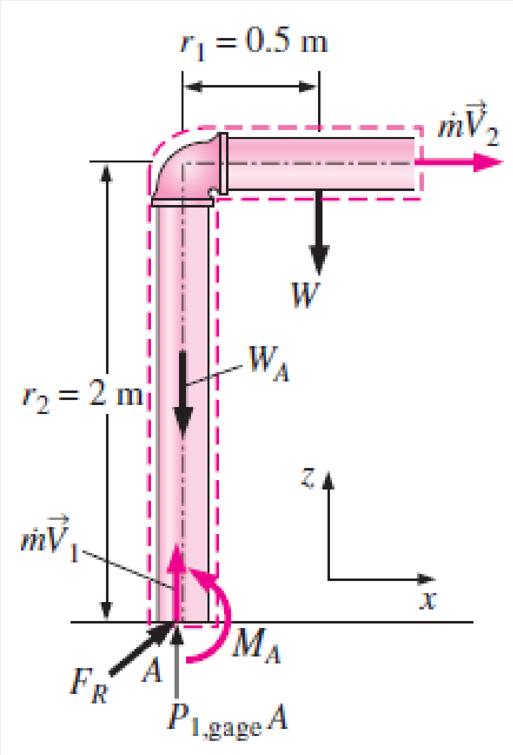
$$\dot{W}_{\text{shaft}} = \omega T_{\text{shaft}} = 2\pi\dot{n}T_{\text{shaft}}$$ $$\omega = 2\pi\dot{n}$$

An annular control volume that encloses the impeller section of a centrifugal pump.

EXAMPLE 6–8 Bending Moment Acting at the Base of a Water Pipe

Underground water is pumped through a 10-cm-diameter pipe that consists of a 2-m-long vertical and 1-m-long horizontal section, as shown in Fig. 6–36. Water discharges to atmospheric air at an average velocity of 3 m/s, and the mass of the horizontal pipe section when filled with water is 12 kg per meter length. The pipe is anchored on the ground by a concrete base. Determine the bending moment acting at the base of the pipe (point *A*) and the required length of the horizontal section that would make the moment at point *A* zero.





Analysis We take the entire L-shaped pipe as the control volume, and designate the inlet by 1 and the outlet by 2. We also take the *x*- and *z*-coordinates as shown. The control volume and the reference frame are fixed.

The conservation of mass equation for this one-inlet, one-outlet, steady-flow system is $\dot{m}_1 = \dot{m}_2 = \dot{m}$, and $V_1 = V_2 = V$ since A_c = constant. The mass flow rate and the weight of the horizontal section of the pipe are

$$\dot{m} = \rho A_c V = (1000 \text{ kg/m}^3)[\pi(0.10 \text{ m})^2/4](3 \text{ m/s}) = 23.56 \text{ kg/s}$$

$$W = mg = (12 \text{ kg/m})(1 \text{ m})(9.81 \text{ m/s}^2)\left(\frac{1 \text{ N}}{1 \text{ kg} \cdot \text{m/s}^2}\right) = 117.7 \text{ N}$$

To determine the moment acting on the pipe at point *A*, we need to take the moment of all forces and momentum flows about that point. This is a steady-flow problem, and all forces and momentum flows are in the same plane. Therefore, the angular momentum equation in this case is expressed as

$$\sum M = \sum_{\text{out}} r\dot{m}V - \sum_{\text{in}} r\dot{m}V$$

where *r* is the average moment arm, *V* is the average speed, all moments in the counterclockwise direction are positive, and all moments in the clockwise direction are negative.

The free-body diagram of the L-shaped pipe is given in Fig. 6–36. Noting that the moments of all forces and momentum flows passing through point *A* are zero, the only force that yields a moment about point *A* is the weight *W* of the horizontal pipe section, and the only momentum flow that yields a moment is the outlet stream (both are negative since both moments are in the clockwise direction). Then the angular momentum equation about point *A* becomes

$$M_A - r_1 W = -r_2 \dot{m} V_2$$

Solving for M_A and substituting give

$$M_A = r_1 W - r_2 \dot{m} V_2$$

$$= (0.5\ \text{m})(118\ \text{N}) - (2\ \text{m})(23.56\ \text{kg/s})(3\ \text{m/s})\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right)$$

$$= -82.5\ \text{N}\cdot\text{m}$$

The negative sign indicates that the assumed direction for M_A is wrong and should be reversed. Therefore, a moment of 82.5 N · m acts at the stem of the pipe in the clockwise direction. That is, the concrete base must apply a 82.5 N · m moment on the pipe stem in the clockwise direction to counteract the excess moment caused by the exit stream.

The weight of the horizontal pipe is $w = W/L = 117.7$ N per m length. Therefore, the weight for a length of L m is Lw with a moment arm of $r_1 = L/2$. Setting $M_A = 0$ and substituting, the length L of the horizontal pipe that would cause the moment at the pipe stem to vanish is determined to be

$$0 = r_1 W - r_2 \dot{m} V_2 \quad \rightarrow \quad 0 = (L/2)Lw - r_2 \dot{m} V_2$$

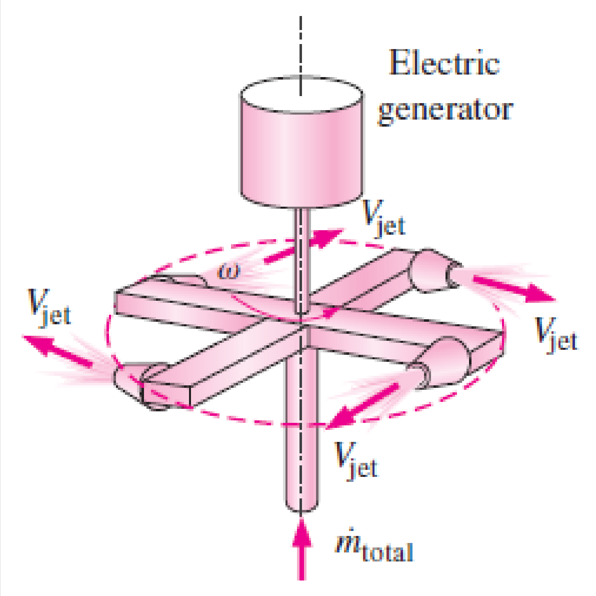
or

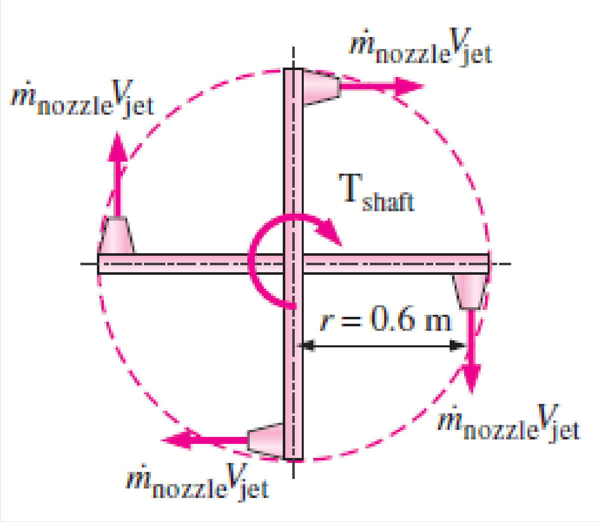
$$L = \sqrt{\frac{2r_2 \dot{m} V_2}{w}} = \sqrt{\frac{2(2\ \text{m})(23.56\ \text{kg/s})(3\ \text{m/s})}{117.7\ \text{N/m}}\left(\frac{\text{N}}{\text{kg}\cdot\text{m/s}^2}\right)} = 1.55\ \text{m}$$

Discussion Note that the pipe weight and the momentum of the exit stream cause opposing moments at point A. This example shows the importance of accounting for the moments of momentums of flow streams when performing a dynamic analysis and evaluating the stresses in pipe materials at critical cross sections.

EXAMPLE 6–9 **Power Generation from a Sprinkler System**

A large lawn sprinkler with four identical arms is to be converted into a turbine to generate electric power by attaching a generator to its rotating head, as shown in Fig. 6–37. Water enters the sprinkler from the base along the axis of rotation at a rate of 20 L/s and leaves the nozzles in the tangential direction. The sprinkler rotates at a rate of 300 rpm in a horizontal plane. The diameter of each jet is 1 cm, and the normal distance between the axis of rotation and the center of each nozzle is 0.6 m. Estimate the electric power produced.





Analysis We take the disk that encloses the sprinkler arms as the control volume, which is a stationary control volume.

The conservation of mass equation for this steady-flow system is $\dot{m}_1 = \dot{m}_2 = \dot{m}_{total}$. Noting that the four nozzles are identical, we have $\dot{m}_{nozzle} = \dot{m}_{total}/4$ or $\dot{V}_{nozzle} = \dot{V}_{total}/4$ since the density of water is constant. The average jet exit velocity relative to the rotating nozzle is

$$V_{jet,r} = \frac{\dot{V}_{nozzle}}{A_{jet}} = \frac{5 \text{ L/s}}{[\pi(0.01 \text{ m})^2/4]}\left(\frac{1 \text{ m}^3}{1000 \text{ L}}\right) = 63.66 \text{ m/s}$$

The angular and tangential velocities of the nozzles are

$$\omega = 2\pi\dot{n} = 2\pi(300 \text{ rev/min})\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 31.42 \text{ rad/s}$$

$$V_{nozzle} = r\omega = (0.6 \text{ m})(31.42 \text{ rad/s}) = 18.85 \text{ m/s}V$$

Note that water in the nozzle is also moving at an average velocity of 18.85 m/s in the opposite direction when it is discharged. The average absolute velocity of the water jet (velocity relative to a fixed location on earth) is the vector sum of its relative velocity (jet velocity relative to the nozzle) and the absolute nozzle velocity,

$$\vec{V}_{jet} = \vec{V}_{jet,r} + \vec{V}_{nozzle}$$

All of these three velocities are in the tangential direction, and taking the direction of jet flow as positive, the vector equation can be written in scalar form using magnitudes as

$$V_{jet} = V_{jet,r} - V_{nozzle} = 63.66 - 18.85 = 44.81 \text{ m/s}$$

Noting that this is a cyclically steady-flow problem, and all forces and momentum flows are in the same plane, the angular momentum equation is approximated as $\sum M = \sum_{\text{out}} r\dot{m}V - \sum_{\text{in}} r\dot{m}V$, where r is the moment arm, all moments in the counterclockwise direction are positive, and all moments in the clockwise direction are negative.

The free-body diagram of the disk that contains the sprinkler arms is given in Fig. 6–37. Note that the moments of all forces and momentum flows passing through the axis of rotation are zero. The momentum flows via the water jets leaving the nozzles yield a moment in the clockwise direction and the effect of the generator on the control volume is a moment also in the clockwise direction (thus both are negative). Then the angular momentum equation about the axis of rotation becomes

$$-\mathrm{T}_{\text{shaft}} = -4r\dot{m}_{\text{nozzle}}V_{\text{jet}} \qquad \text{or} \qquad \mathrm{T}_{\text{shaft}} = r\dot{m}_{\text{total}}V_{\text{jet}}$$

Substituting, the torque transmitted through the shaft is

$$\mathrm{T}_{\text{shaft}} = r\dot{m}_{\text{total}}V_{\text{jet}} = (0.6\ \text{m})(20\ \text{kg/s})(44.81\ \text{m/s})\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right) = 537.7\ \text{N}\cdot\text{m}$$

since $\dot{m}_{\text{total}} = \rho\dot{V}_{\text{total}} = (1\ \text{kg/L})(20\ \text{L/s}) = 20\ \text{kg/s}$.

Then the power generated becomes

$$\dot{W} = 2\pi\dot{n}\mathrm{T}_{\text{shaft}} = \omega\mathrm{T}_{\text{shaft}} = (31.42\ \text{rad/s})(537.7\ \text{N}\cdot\text{m})\left(\frac{1\ \text{kW}}{1000\ \text{N}\cdot\text{m/s}}\right) = \mathbf{16.9\ kW}$$

Therefore, this sprinkler-type turbine has the potential to produce 16.9 kW of power.

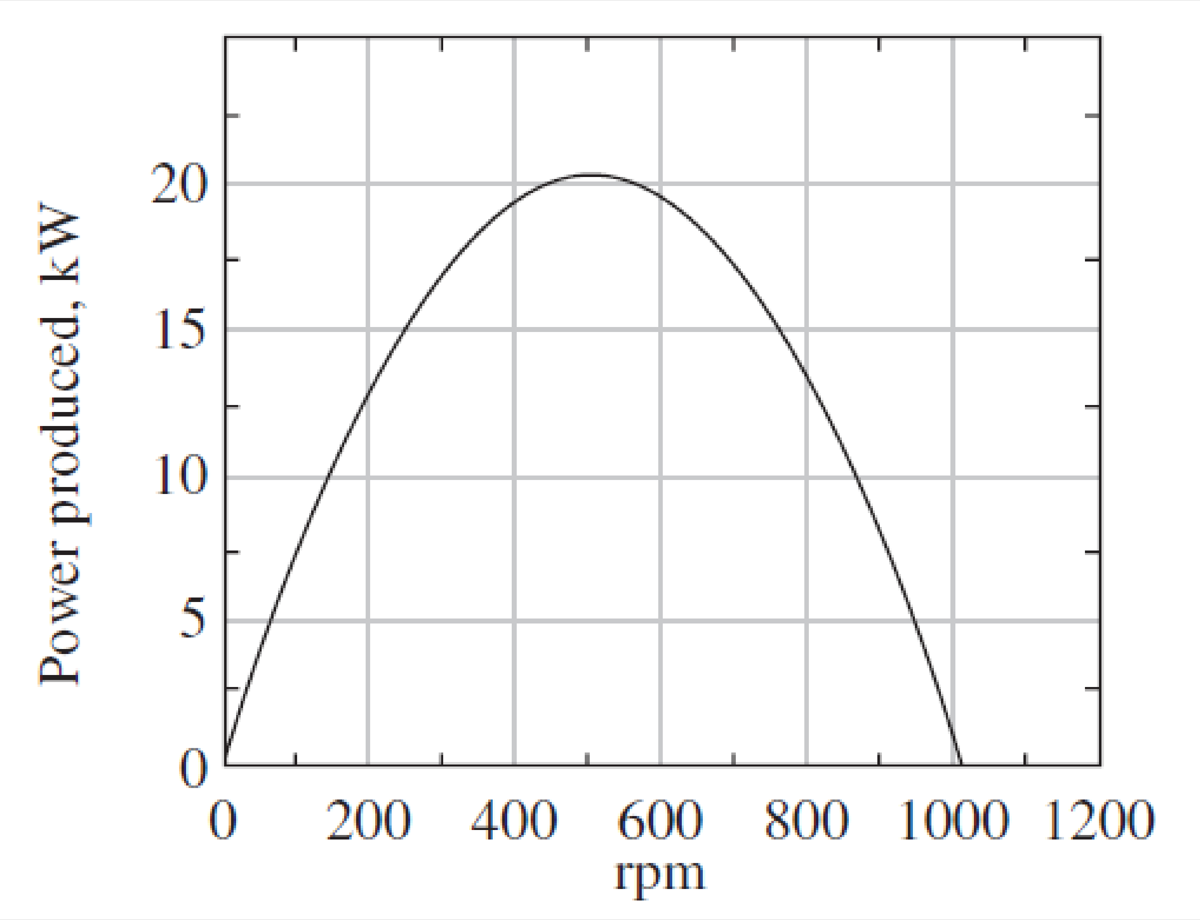
Discussion To put the result obtained in perspective, we consider two limiting cases. In the first limiting case, the sprinkler is stuck, and thus, the angular velocity is zero. The torque developed is maximum in this case, since $V_{nozzle} = 0$. Thus $V_{jet} = V_{jet,\,r} = 63.66$ m/s, giving $T_{shaft,\,max} = 764$ N · m. The power generated is zero since the generator shaft does not rotate.

In the second limiting case, the sprinkler shaft is disconnected from the generator (and thus both the useful torque and power generation are zero), and the shaft accelerates until it reaches an equilibrium velocity. Setting $T_{shaft} = 0$ in the angular momentum equation gives the absolute water-jet velocity (jet velocity relative to an observer on earth) to be zero, $V_{jet} = 0$. Therefore, the relative velocity $V_{jet,\,r}$ and absolute velocity V_{nozzle} are equal but in opposite direction. So, the absolute tangential velocity of the jet (and thus torque) is zero, and the water mass drops straight down like a waterfall under gravity with zero angular momentum (around the axis of rotation). The angular speed of the sprinkler in this case is

$$\dot{n} = \frac{\omega}{2\pi} = \frac{V_{nozzle}}{2\pi r} = \frac{63.66 \text{ m/s}}{2\pi(0.6 \text{ m})}\left(\frac{60 \text{ s}}{1 \text{ min}}\right) = 1013 \text{ rpm}$$

Of course, the $T_{shaft} = 0$ case is possible only for an ideal, frictionless nozzle (i.e., 100 percent nozzle efficiency, as a no-load ideal turbine). Otherwise, there would be a resisting torque due to friction of the water, shaft, and surrounding air.

The variation of power produced with angular speed is plotted in Fig. 6–38. Note that the power produced increases with increasing rpm, reaches a maximum (at about 500 rpm in this case), and then decreases. The actual power produced would be less than this due to generator inefficiency (Chap. 5) and other irreversible losses such as fluid friction within the nozzle (Chap. 8), shaft friction, and aerodynamic drag (Chap. 11).



The variation of power produced with angular speed for the turbine of Example 6–9.

Summary

- Newton's Laws
- Choosing a Control Volume
- Forces Acting on a Control Volume
- The Linear Momentum Equation
  - ✓ Special Cases
  - ✓ Momentum-Flux Correction Factor, β
  - ✓ Steady Flow
  - ✓ Flow with No External Forces
- Review of Rotational Motion and Angular Momentum
- The Angular Momentum Equation
  - ✓ Special Cases
  - ✓ Flow with No External Moments
  - ✓ Radial-Flow Devices