Heat and Mass Transfer: Fundamentals & Applications

5th Edition in SI Units

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Chapter 2

HEAT CONDUCTION EQUATION

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Objectives

- Understand multidimensionality and time dependence of heat transfer, and the conditions under which a heat transfer problem can be approximated as being one-dimensional.
- Obtain the differential equation of heat conduction in various coordinate systems, and simplify it for steady one-dimensional case.
- Identify the thermal conditions on surfaces, and express them mathematically as boundary and initial conditions.
- Solve one-dimensional heat conduction problems and obtain the temperature distributions within a medium and the heat flux.
- Analyze one-dimensional heat conduction in solids that involve heat generation.
- Evaluate heat conduction in solids with temperature-dependent thermal conductivity.

INTRODUCTION

- Although heat transfer and temperature are closely related, they are of a different nature.
- Temperature has only magnitude. It is a *scalar* quantity.
- Heat transfer has direction as well as magnitude. It is a *vector* quantity.
- We work with a coordinate system and indicate direction with plus or minus signs.

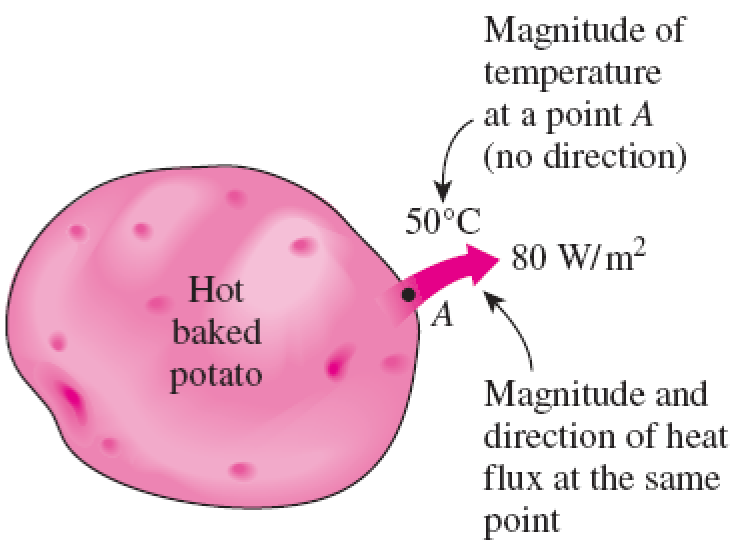


FIGURE 2–1
Heat transfer has direction as well as magnitude, and thus it is a *vector* quantity.

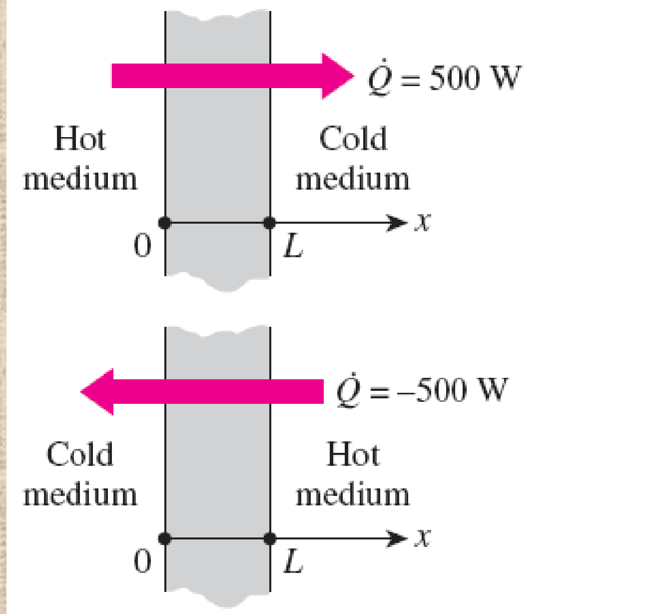
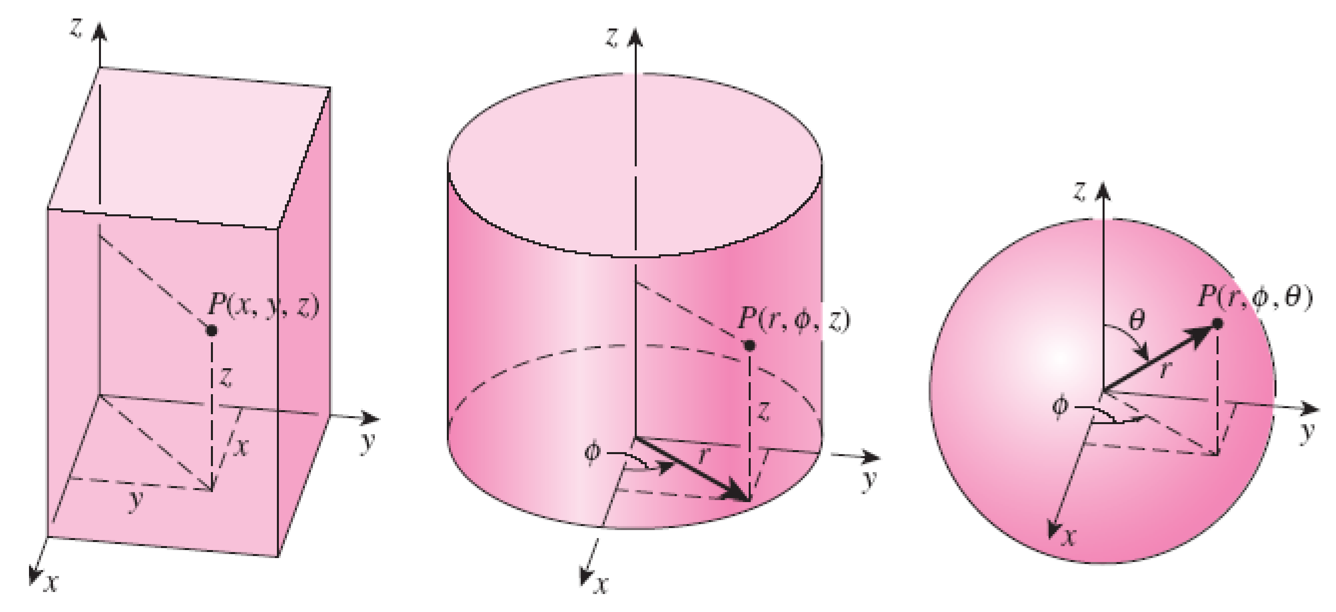


FIGURE 2–2
Indicating direction for heat transfer (positive in the positive direction; negative in the negative direction).

- The driving force for any form of heat transfer is the *temperature difference.*
- The larger the temperature difference, the larger the rate of heat transfer.
- Three prime coordinate systems:
  - ✓ rectangular $T(x, y, z, t)$
  - ✓ cylindrical $T(r, \phi, z, t)$
  - ✓ spherical $T(r, \phi, \theta, t)$.

FIGURE 2–3
The various distances and angles involved when describing the location of a point in different coordinate systems.



(*a*) Rectangular coordinates (*b*) Cylindrical coordinates (*c*) Spherical coordinates

Steady versus Transient Heat Transfer

- ***Steady*** implies *no change* with time at any point within the medium
- ***Transient*** implies *variation with time* or *time dependence*
- In the special case of variation with time but not with position, the temperature of the medium changes *uniformly* with time. Such heat transfer systems are called **lumped systems**.

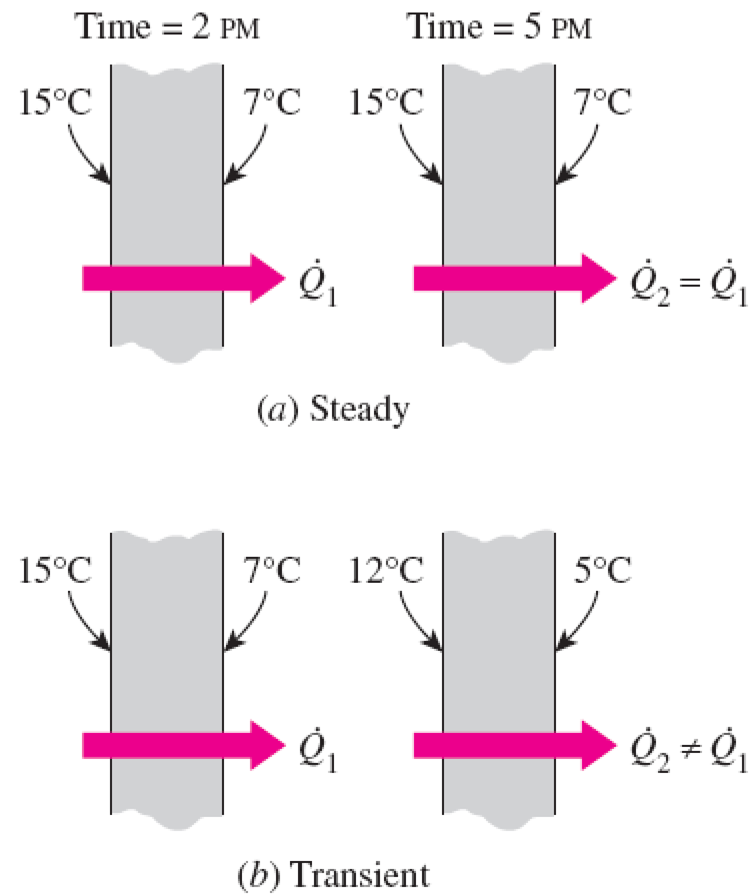


FIGURE 2–4

Transient and steady heat conduction in a plane wall.

Multidimensional Heat Transfer

- Heat transfer problems are also classified as being:
  - ✓ ***one-dimensional***
  - ✓ ***two dimensional***
  - ✓ ***three-dimensional***
- In the most general case, heat transfer through a medium is **three-dimensional**. However, some problems can be classified as two- or one-dimensional depending on the relative magnitudes of heat transfer rates in different directions and the level of accuracy desired.
- **One-dimensional** if the temperature in the medium varies in one direction only and thus heat is transferred in one direction, and the variation of temperature and thus heat transfer in other directions are negligible or zero.
- **Two-dimensional** if the temperature in a medium, in some cases, varies mainly in two primary directions, and the variation of temperature in the third direction (and thus heat transfer in that direction) is negligible.

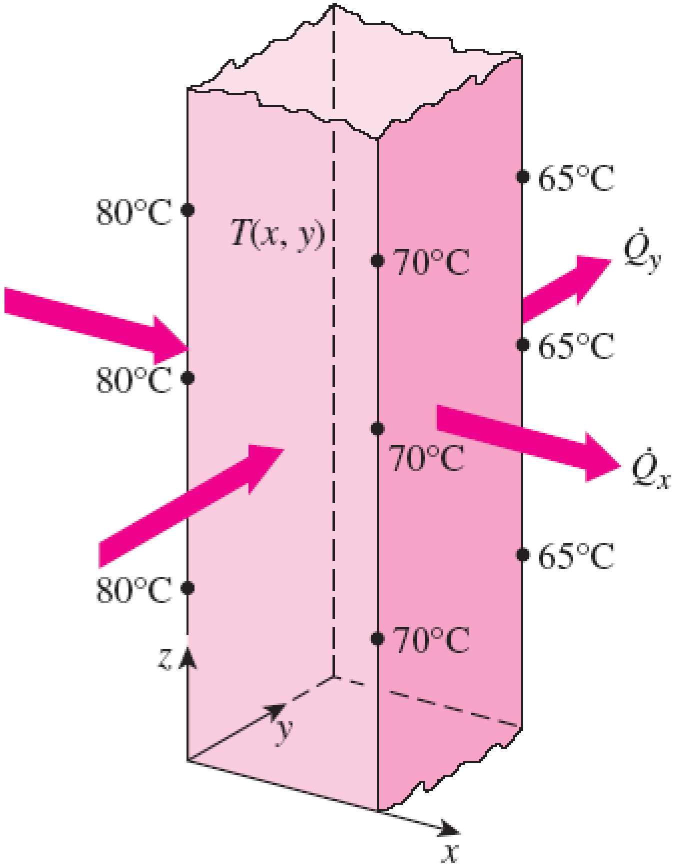


FIGURE 2–5

Two-dimensional heat transfer in a long rectangular bar.

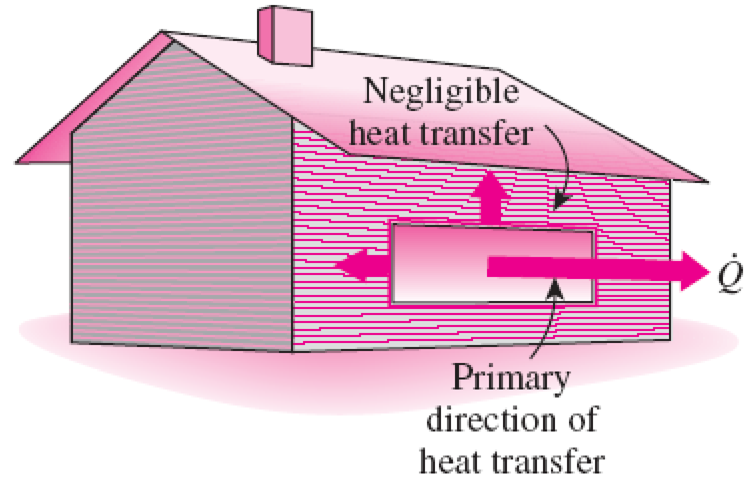


FIGURE 2–6

Heat transfer through the window of a house can be taken to be one-dimensional.

- The rate of heat conduction through a medium in a specified direction (say, in the *x*-direction) is expressed by **Fourier's law of heat conduction** for one-dimensional heat conduction as:

$$\dot{Q}_{\text{cond}} = -kA\frac{dT}{dx} \qquad \text{(W)}$$

Heat is conducted in the direction of decreasing temperature, and thus the temperature gradient is negative when heat is conducted in the positive *x* -direction.

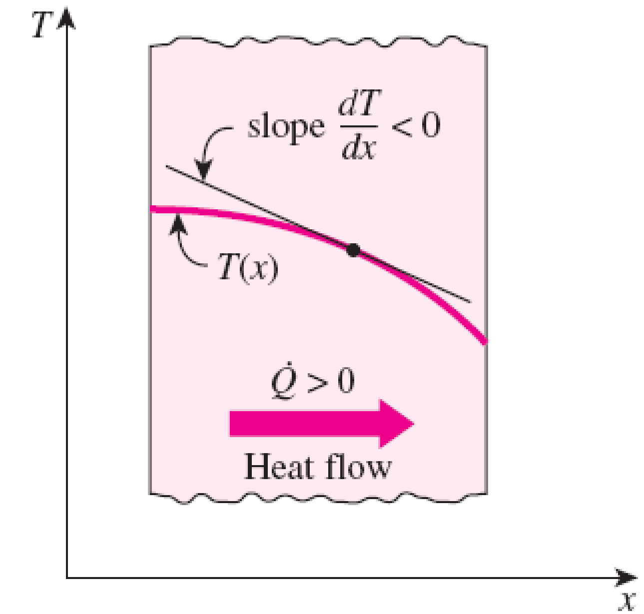


FIGURE 2–7

The temperature gradient dT/dx is simply the slope of the temperature curve on a T-x diagram.

- The heat flux vector at a point *P* on the surface of the figure must be perpendicular to the surface, and it must point in the direction of decreasing temperature
- If *n* is the normal of the isothermal surface at point *P*, the rate of heat conduction at that point can be expressed by Fourier's law as

$$\dot{Q}_n = -kA\frac{\partial T}{\partial n} \qquad \text{(W)}$$

$$\vec{\dot{Q}}_n = \dot{Q}_x \vec{i} + \dot{Q}_y \vec{j} + \dot{Q}_z \vec{k}$$

$$\dot{Q}_x = -kA_x \frac{\partial T}{\partial x}, \qquad \dot{Q}_y = -kA_y \frac{\partial T}{\partial y},$$

$$\dot{Q}_z = -kA_z \frac{\partial T}{\partial z}$$

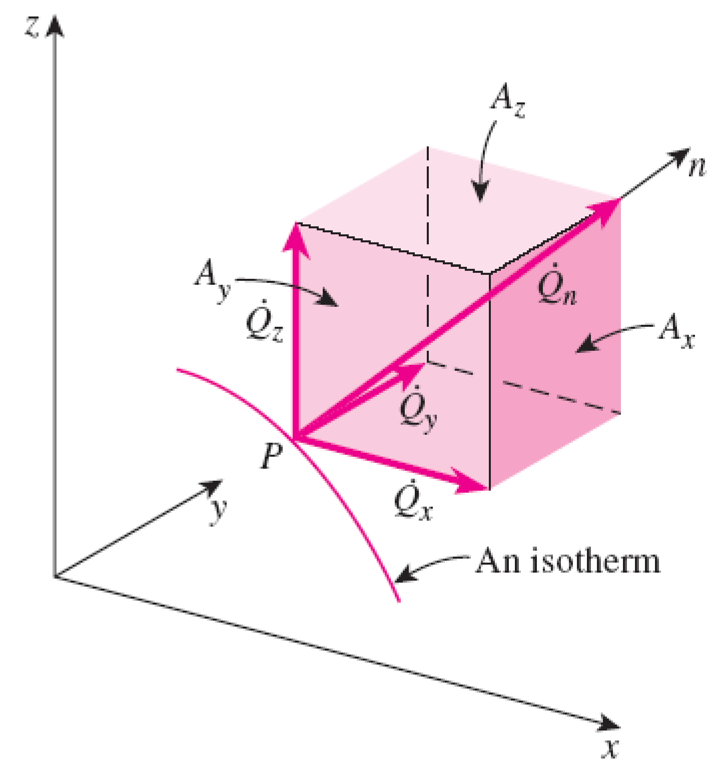


FIGURE 2–8
The heat transfer vector is always normal to an isothermal surface and can be resolved into its components like any other vector.

Heat Generation

- Examples:
  - ✓ electrical energy being converted to heat at a rate of I^2R,
  - ✓ fuel elements of nuclear reactors,
  - ✓ exothermic chemical reactions.
- Heat generation is a *volumetric phenomenon.*
- The rate of heat generation units : W/m^3 or Btu/h·ft^3.
- The rate of heat generation in a medium may vary with time as well as position within the medium.

$$\dot{E}_{gen} = \int_V \dot{e}_{gen} dV \qquad \text{(W)}$$

$$\dot{E}_{gen} = \dot{e}_{gen} V,$$

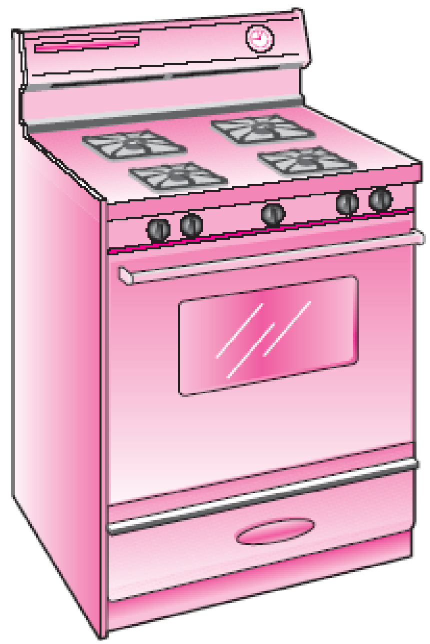


FIGURE 2–9

Heat is generated in the heating coils of an electric range as a result of the conversion of electrical energy to heat.

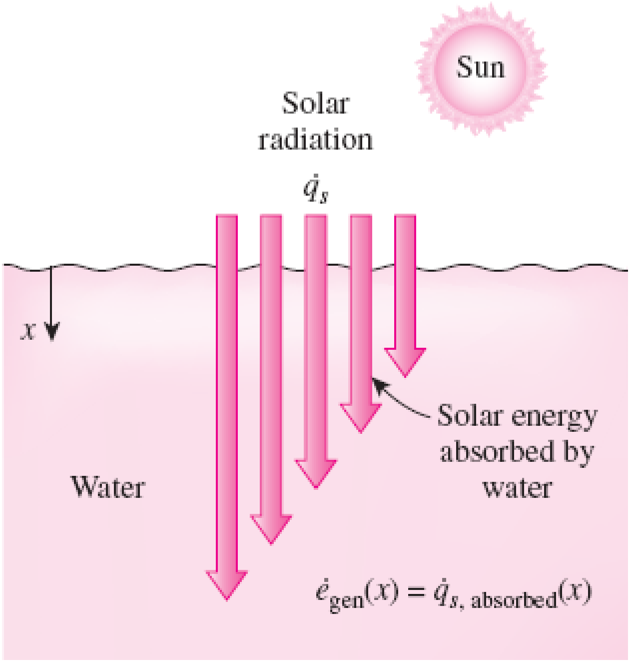


FIGURE 2–10

The absorption of solar radiation by water can be treated as heat generation.

EXAMPLE 2–1 Heat Generation in a Hair Dryer

The resistance wire of a 1200-W hair dryer is 80 cm long and has a diameter of $D = 0.3$ cm (Fig. 2–11). Determine the rate of heat generation in the wire per unit volume, in W/cm^3, and the heat flux on the outer surface of the wire as a result of this heat generation.

FIGURE 2–11
Schematic for Example 2–1.

SOLUTION The power consumed by the resistance wire of a hair dryer is given. The heat generation and the heat flux are to be determined.

Assumptions Heat is generated uniformly in the resistance wire.

Analysis A 1200-W hair dryer converts electrical energy into heat in the wire at a rate of 1200 W. Therefore, the rate of heat generation in a resistance wire is equal to the power consumption of a resistance heater. Then the rate of heat generation in the wire per unit volume is determined by dividing the total rate of heat generation by the volume of the wire,

$$\dot{e}_{\text{gen}} = \frac{\dot{E}_{\text{gen}}}{V_{\text{wire}}} = \frac{\dot{E}_{\text{gen}}}{(\pi D^2/4)L} = \frac{1200 \text{ W}}{[\pi(0.3 \text{ cm})^2/4](80 \text{ cm})} = 212 \text{ W/cm}^3$$

Similarly, heat flux on the outer surface of the wire as a result of this heat generation is determined by dividing the total rate of heat generation by the surface area of the wire,

$$\dot{Q}_s = \frac{\dot{E}_{\text{gen}}}{A_{\text{wire}}} = \frac{\dot{E}_{\text{gen}}}{\pi DL} = \frac{1200 \text{ W}}{\pi(0.3 \text{ cm})(80 \text{ cm})} = 15.9 \text{ W/cm}^2$$

Discussion Note that heat generation is expressed per unit volume in W/cm^3 or Btu/h·ft^3, whereas heat flux is expressed per unit surface area in W/cm^2 or Btu/h·ft^2.

ONE-DIMENSIONAL HEAT CONDUCTION EQUATION

Consider heat conduction through a large plane wall such as the wall of a house, the glass of a single pane window, the metal plate at the bottom of a pressing iron, a cast-iron steam pipe, a cylindrical nuclear fuel element, an electrical resistance wire, the wall of a spherical container, or a spherical metal ball that is being quenched or tempered.

Heat conduction in these and many other geometries can be approximated as being *one-dimensional* since heat conduction through these geometries is dominant in one direction and negligible in other directions.

Next we develop the onedimensional heat conduction equation in rectangular, cylindrical, and spherical coordinates.

Heat Conduction Equation in a Large Plane Wall

$$\begin{pmatrix}\text{Rate of heat}\\\text{conduction}\\\text{at } x\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\\text{conduction}\\\text{at } x+\Delta x\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\\text{generation}\\\text{inside the}\\\text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\\text{of the energy}\\\text{content of the}\\\text{element}\end{pmatrix}$$

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{E}_{\text{gen, element}} = \frac{\Delta E_{\text{element}}}{\Delta t} \qquad \textbf{(2-6)}$$

$$\Delta E_{\text{element}} = E_{t+\Delta t} - E_t = mc(T_{t+\Delta t} - T_t) = \rho c A \Delta x (T_{t+\Delta t} - T_t)$$

$$\dot{E}_{\text{gen, element}} = \dot{e}_{\text{gen}} V_{\text{element}} = \dot{e}_{\text{gen}} A \Delta x$$

Substituting into Eq. 2–6, we get

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{e}_{\text{gen}} A \Delta x = \rho c A \Delta x \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Dividing by $A\Delta x$ gives

$$-\frac{1}{A}\frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} + \dot{e}_{\text{gen}} = \rho c \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Taking the limit as $\Delta x \to 0$ and $\Delta t \to 0$ yields

$$\frac{1}{A}\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) + \dot{e}_{\text{gen}} = \rho c \frac{\partial T}{\partial t}$$

Since $$\lim_{\Delta x \to 0} \frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} = \frac{\partial \dot{Q}}{\partial x} = \frac{\partial}{\partial x}\left(-kA\frac{\partial T}{\partial x}\right)$$

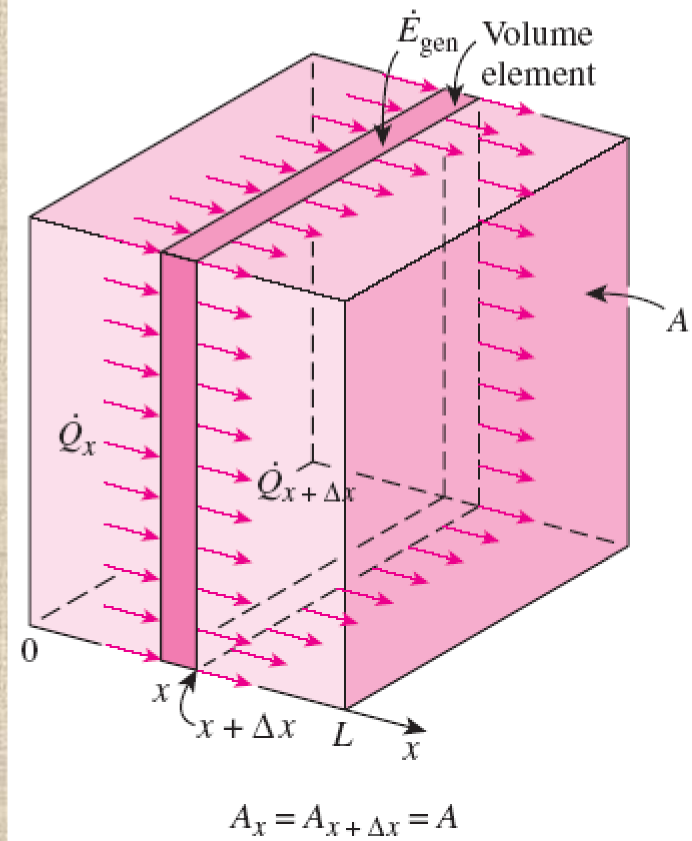


FIGURE 2–12

One-dimensional heat conduction through a volume element in a large plane wall.

Heat Conduction Equation in a Large Plane Wall

Variable conductivity: $$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{e}_{\text{gen}} = \rho c \frac{\partial T}{\partial t}$$

Constant conductivity: $$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{e}_{\text{gen}}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(1) *Steady-state:* $(\partial/\partial t = 0)$
$$\frac{d^2 T}{dx^2} + \frac{\dot{e}_{\text{gen}}}{k} = 0$$

(2) *Transient, no heat generation:* $(\dot{e}_{\text{gen}} = 0)$
$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(3) *Steady-state, no heat generation:* $(\partial/\partial t = 0 \text{ and } \dot{e}_{\text{gen}} = 0)$
$$\frac{d^2 T}{dx^2} = 0$$

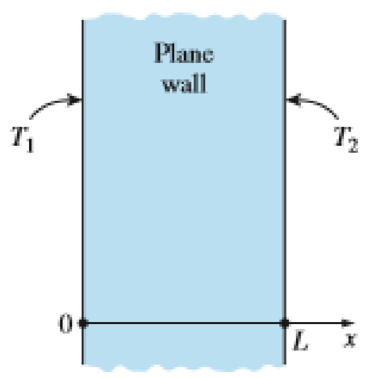
General, one-dimensional:

No generation | Steady-state

$$\frac{\partial^2 T}{\partial x^2} + \overset{0}{\cancel{\frac{\dot{e}_{\text{gen}}}{k}}} = \overset{0}{\cancel{\frac{1}{\alpha}\frac{\partial T}{\partial t}}}$$

Steady, one-dimensional:

$$\frac{d^2 T}{dx^2} = 0$$



The simplification of the one-dimensional heat conduction equation in a plane wall for the case of constant conductivity for steady conduction with no heat generation.

Heat Conduction Equation in a Long Cylinder

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } r\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } r+\Delta r\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of the}\\ \text{element}\end{pmatrix}$$

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{E}_{\text{gen, element}} = \frac{\Delta E_{\text{element}}}{\Delta t}$$

$$\Delta E_{\text{element}} = E_{t+\Delta t} - E_t = mc(T_{t+\Delta t} - T_t) = \rho c A \Delta r (T_{t+\Delta t} - T_t)$$

$$\dot{E}_{\text{gen, element}} = \dot{e}_{\text{gen}} V_{\text{element}} = \dot{e}_{\text{gen}} A \Delta r$$

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{e}_{\text{gen}} A \Delta r = \rho c A \Delta r \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

$$-\frac{1}{A}\frac{\dot{Q}_{r+\Delta r} - \dot{Q}_r}{\Delta r} + \dot{e}_{\text{gen}} = \rho c \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Taking the limit as $\Delta r \to 0$ and $\Delta t \to 0$ yields

$$\frac{1}{A}\frac{\partial}{\partial r}\left(kA\frac{\partial T}{\partial r}\right) + \dot{e}_{\text{gen}} = \rho c \frac{\partial T}{\partial t}$$

$$\lim_{\Delta r \to 0} \frac{\dot{Q}_{r+\Delta r} - \dot{Q}_r}{\Delta r} = \frac{\partial \dot{Q}}{\partial r} = \frac{\partial}{\partial r}\left(-kA\frac{\partial T}{\partial r}\right)$$

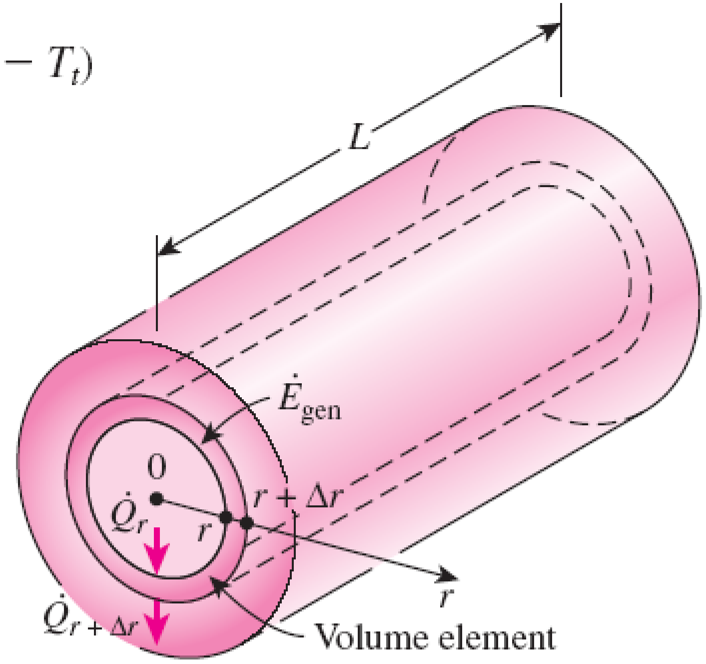


FIGURE 2–14

One-dimensional heat conduction through a volume element in a long cylinder.

Heat Conduction Equation in a Long Cylinder

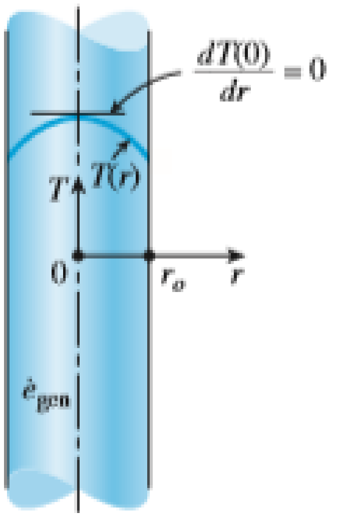
Variable conductivity: $$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + \dot{e}_{gen} = \rho c\frac{\partial T}{\partial t}$$

Constant conductivity: $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\dot{e}_{gen}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(1) *Steady-state:* $(\partial/\partial t = 0)$ $$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{e}_{gen}}{k} = 0$$

(2) *Transient, no heat generation:* $(\dot{e}_{gen} = 0)$ $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(3) *Steady-state, no heat generation:* $(\partial/\partial t = 0 \text{ and } \dot{e}_{gen} = 0)$ $$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$



(*a*) The form that is ready to integrate

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

(*b*) The equivalent alternative form

$$r\frac{d^2T}{dr^2} + \frac{dT}{dr} = 0$$

Two equivalent forms of the differential equation for the one-dimensional steady heat conduction in a cylinder with no heat generation.

Heat Conduction Equation in a Sphere

Variable conductivity:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k \frac{\partial T}{\partial r}\right) + \dot{e}_{\text{gen}} = \rho c \frac{\partial T}{\partial t}$$

Constant conductivity:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{\dot{e}_{\text{gen}}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

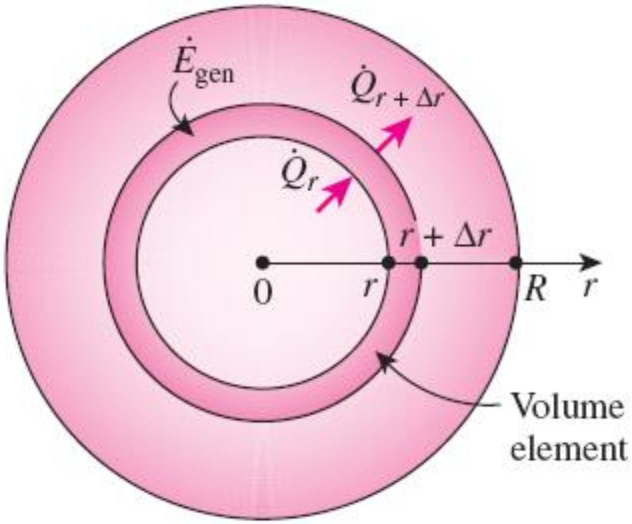


FIGURE 2–16

One-dimensional heat conduction through a volume element in a sphere.

(1) *Steady-state:* $(\partial/\partial t = 0)$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) + \frac{\dot{e}_{\text{gen}}}{k} = 0$$

(2) *Transient, no heat generation:* $(\dot{e}_{\text{gen}} = 0)$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(3) *Steady-state, no heat generation:* $(\partial/\partial t = 0$ and $\dot{e}_{\text{gen}} = 0)$

$$\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0 \qquad \text{or} \qquad r\frac{d^2T}{dr^2} + 2\frac{dT}{dr} = 0$$

Combined One-Dimensional Heat Conduction Equation

An examination of the one-dimensional transient heat conduction equations for the plane wall, cylinder, and sphere reveals that all three equations can be expressed in a compact form as

$$\frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n k \frac{\partial T}{\partial r}\right) + \dot{e}_{\text{gen}} = \rho c \frac{\partial T}{\partial t}$$

$n = 0$ for a plane wall

$n = 1$ for a cylinder

$n = 2$ for a sphere

In the case of a plane wall, it is customary to replace the variable r by x.

This equation can be simplified for steady-state or no heat generation cases as described before.

EXAMPLE 2–2 Heat Conduction through the Bottom of a Pan

Consider a steel pan placed on top of an electric range to cook spaghetti (Fig. 2–17). The bottom section of the pan is 0.4 cm thick and has a diameter of 18 cm. The electric heating unit on the range top consumes 800 W of power during cooking, and 80 percent of the heat generated in the heating element is transferred uniformly to the pan. Assuming constant thermal conductivity, obtain the differential equation that describes the variation of the temperature in the bottom section of the pan during steady operation.

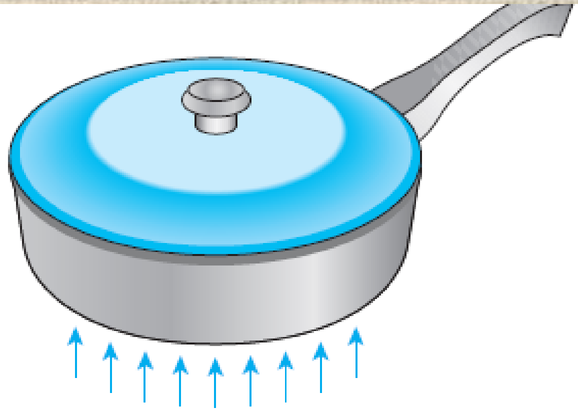


FIGURE 2–17
Schematic for Example 2–2.

SOLUTION A steel pan placed on top of an electric range is considered. The differential equation for the variation of temperature in the bottom of the pan is to be obtained.

Analysis The bottom section of the pan has a large surface area relative to its thickness and can be approximated as a large plane wall. Heat flux is applied to the bottom surface of the pan uniformly, and the conditions on the inner surface are also uniform. Therefore, we expect the heat transfer through the bottom section of the pan to be from the bottom surface toward the top, and heat transfer in this case can reasonably be approximated as being one-dimensional. Taking the direction normal to the bottom surface of the pan to be the *x*-axis, we will have $T = T(x)$ during steady operation since the temperature in this case will depend on *x* only.

The thermal conductivity is given to be constant, and there is no heat generation in the medium (within the bottom section of the pan). Therefore, the differential equation governing the variation of temperature in the bottom section of the pan in this case is simply Eq. 2–17,

$$\frac{d^2T}{dx^2} = 0$$

which is the steady one-dimensional heat conduction equation in rectangular coordinates under the conditions of constant thermal conductivity and no heat generation.

Discussion Note that the conditions at the surface of the medium have no effect on the differential equation.

EXAMPLE 2–3 **Heat Conduction in a Resistance Heater**

A 2-kW resistance heater wire with thermal conductivity $k = 15$ W/m·K, diameter $D = 0.4$ cm, and length $L = 50$ cm is used to boil water by immersing it in water (Fig. 2–18). Assuming the variation of the thermal conductivity of the wire with temperature to be negligible, obtain the differential equation that describes the variation of the temperature in the wire during steady operation.

SOLUTION The resistance wire of a water heater is considered. The differential equation for the variation of temperature in the wire is to be obtained.

Analysis The resistance wire can be considered to be a very long cylinder since its length is more than 100 times its diameter. Also, heat is generated uniformly in the wire and the conditions on the outer surface of the wire are uniform. Therefore, it is reasonable to expect the temperature in the wire to vary in the radial r direction only and thus the heat transfer to be one-dimensional. Then we have $T = T(r)$ during steady operation since the temperature in this case depends on r only.

The rate of heat generation in the wire per unit volume can be determined from

$$\dot{e}_{gen} = \frac{\dot{E}_{gen}}{V_{wire}} = \frac{\dot{E}_{gen}}{(\pi D^2/4)L} = \frac{2000 \text{ W}}{[\pi(0.004 \text{ m})^2/4](0.5 \text{ m})} = 0.318 \times 10^9 \text{ W/m}^3$$

Noting that the thermal conductivity is given to be constant, the differential equation that governs the variation of temperature in the wire is simply Eq. 2–27,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{e}_{gen}}{k} = 0$$

which is the steady one-dimensional heat conduction equation in cylindrical coordinates for the case of constant thermal conductivity.

Discussion Note again that the conditions at the surface of the wire have no effect on the differential equation.

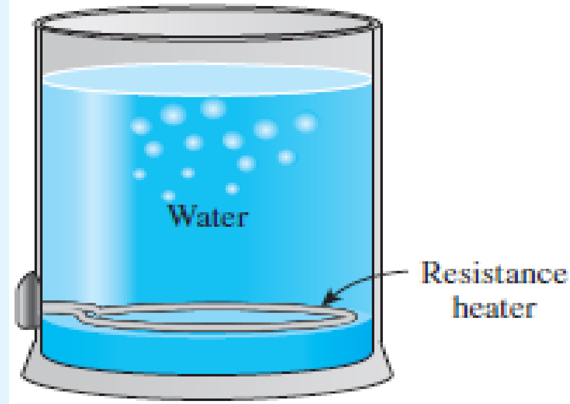


FIGURE 2–18
Schematic for Example 2–3.

EXAMPLE 2–4 Cooling of a Hot Metal Ball in Air

A spherical metal ball of radius *R* is heated in an oven to a temperature of 300°C throughout and is then taken out of the oven and allowed to cool in ambient air at T_∞ = 25°C by convection and radiation (Fig. 2–19). The thermal conductivity of the ball material is known to vary linearly with temperature. Assuming the ball is cooled uniformly from the entire outer surface, obtain the differential equation that describes the variation of the temperature in the ball during cooling.

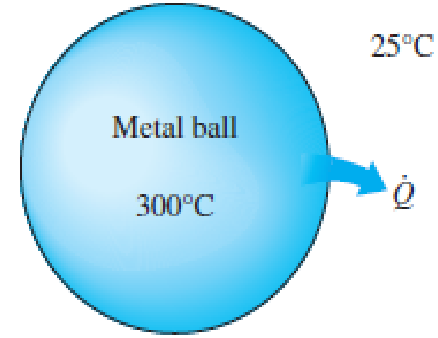


FIGURE 2–19
Schematic for Example 2–4.

SOLUTION A hot metal ball is allowed to cool in ambient air. The differential equation for the variation of temperature within the ball is to be obtained.

Analysis The ball is initially at a uniform temperature and is cooled uniformly from the entire outer surface. Also, the temperature at any point in the ball changes with time during cooling. Therefore, this is a one-dimensional transient heat conduction problem since the temperature within the ball changes with the radial distance *r* and the time *t*. That is, $T = T(r, t)$.

The thermal conductivity is given to be variable, and there is no heat generation in the ball. Therefore, the differential equation that governs the variation of temperature in the ball in this case is obtained from Eq. 2–30 by setting the heat generation term equal to zero. We obtain

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k \frac{\partial T}{\partial r}\right) = \rho c \frac{\partial T}{\partial t}$$

which is the one-dimensional transient heat conduction equation in spherical coordinates under the conditions of variable thermal conductivity and no heat generation.

Discussion Note again that the conditions at the outer surface of the ball have no effect on the differential equation.

GENERAL HEAT CONDUCTION EQUATION

In the last section we considered one-dimensional heat conduction and assumed heat conduction in other directions to be negligible.

Most heat transfer problems encountered in practice can be approximated as being one-dimensional, and we mostly deal with such problems in this text.

However, this is not always the case, and sometimes we need to consider heat transfer in other directions as well.

In such cases heat conduction is said to be *multidimensional*, and in this section we develop the governing differential equation in such systems in rectangular, cylindrical, and spherical coordinate systems.

Rectangular Coordinates

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction at}\\ x, y, \text{and } z\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x+\Delta x,\\ y+\Delta y, \text{and } z+\Delta z\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of}\\ \text{the element}\end{pmatrix}$$

or

$$\dot{Q}_x + \dot{Q}_y + \dot{Q}_z - \dot{Q}_{x+\Delta x} - \dot{Q}_{y+\Delta y} - \dot{Q}_{z+\Delta z} + \dot{E}_{\text{gen, element}} = \frac{\Delta E_{\text{element}}}{\Delta t} \qquad \text{(2–36)}$$

Noting that the volume of the element is $V_{\text{element}} = \Delta x \Delta y \Delta z$, the change in the energy content of the element and the rate of heat generation within the element can be expressed as

$$\Delta E_{\text{element}} = E_{t+\Delta t} - E_t = mc(T_{t+\Delta t} - T_t) = \rho c \Delta x \Delta y \Delta z (T_{t+\Delta t} - T_t)$$
$$\dot{E}_{\text{gen, element}} = \dot{e}_{\text{gen}} V_{\text{element}} = \dot{e}_{\text{gen}} \Delta x \Delta y \Delta z$$

Substituting into Eq. 2–36, we get

$$\dot{Q}_x + \dot{Q}_y + \dot{Q}_z - \dot{Q}_{x+\Delta x} - \dot{Q}_{y+\Delta y} - \dot{Q}_{z+\Delta z} + \dot{e}_{\text{gen}} \Delta x \Delta y \Delta z = \rho c \Delta x \Delta y \Delta z \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Dividing by $\Delta x \Delta y \Delta z$ gives

$$-\frac{1}{\Delta y \Delta z}\frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} - \frac{1}{\Delta x \Delta z}\frac{\dot{Q}_{y+\Delta y} - \dot{Q}_y}{\Delta y} - \frac{1}{\Delta x \Delta y}\frac{\dot{Q}_{z+\Delta z} - \dot{Q}_z}{\Delta z} + \dot{e}_{\text{gen}} =$$
$$\rho c \frac{T_{t+\Delta t} - T_t}{\Delta t} \qquad \text{(2–37)}$$

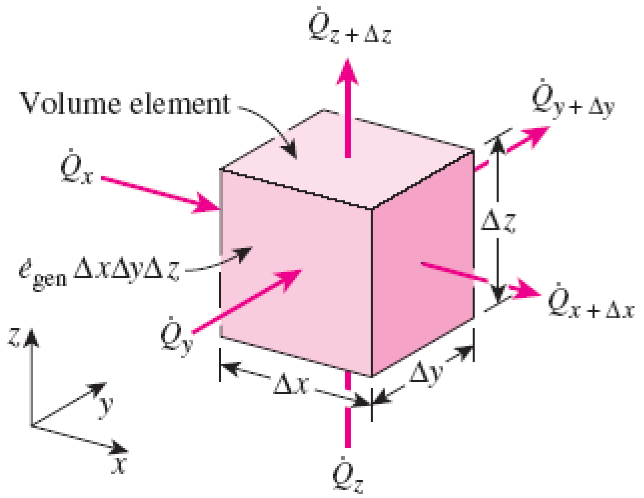


FIGURE 2–20

Three-dimensional heat conduction through a rectangular volume element.

Noting that the heat transfer areas of the element for heat conduction in the x, y, and z directions are $A_x = \Delta y \Delta z$, $A_y = \Delta x \Delta z$, and $A_z = \Delta x \Delta y$, respectively, and taking the limit as Δx, Δy, Δz and $\Delta t \to 0$ yields

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{e}_{\text{gen}} = \rho c \frac{\partial T}{\partial t} \qquad \textbf{(2–38)}$$

since, from the definition of the derivative and Fourier's law of heat conduction,

$$\lim_{\Delta x \to 0} \frac{1}{\Delta y \Delta z}\frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} = \frac{1}{\Delta y \Delta z}\frac{\partial Q_x}{\partial x} = \frac{1}{\Delta y \Delta z}\frac{\partial}{\partial x}\left(-k\Delta y \Delta z\frac{\partial T}{\partial x}\right) = -\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)$$

$$\lim_{\Delta y \to 0} \frac{1}{\Delta x \Delta z}\frac{\dot{Q}_{y+\Delta y} - \dot{Q}_y}{\Delta y} = \frac{1}{\Delta x \Delta z}\frac{\partial Q_y}{\partial y} = \frac{1}{\Delta x \Delta z}\frac{\partial}{\partial y}\left(-k\Delta x \Delta z\frac{\partial T}{\partial y}\right) = -\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)$$

$$\lim_{\Delta z \to 0} \frac{1}{\Delta x \Delta y}\frac{\dot{Q}_{z+\Delta z} - \dot{Q}_z}{\Delta z} = \frac{1}{\Delta x \Delta y}\frac{\partial Q_z}{\partial z} = \frac{1}{\Delta x \Delta y}\frac{\partial}{\partial z}\left(-k\Delta x \Delta y\frac{\partial T}{\partial z}\right) = -\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$

Eq. 2–38 is the general heat conduction equation in rectangular coordinates. In the case of constant thermal conductivity, it reduces to

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{e}_{\text{gen}}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \textbf{(2–39)}$$

where the property $\alpha = k/\rho c$ is again the *thermal diffusivity* of the material. Eq. 2–39 is known as the **Fourier-Biot equation**, and it reduces to these forms under specified conditions:

(1) *Steady-state:*
(called the **Poisson equation**)

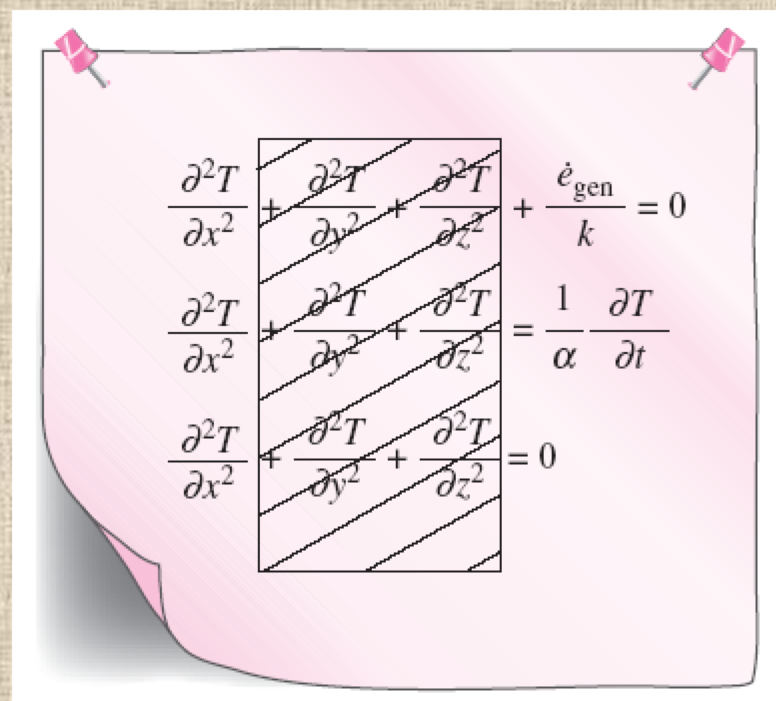
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{e}_{\text{gen}}}{k} = 0$$

(2) *Transient, no heat generation:*
(called the **diffusion equation**)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(3) *Steady-state, no heat generation:*
(called the **Laplace equation**)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$



The three-dimensional heat conduction equations reduce to the one-dimensional ones when the temperature varies in one dimension only.

Cylindrical Coordinates

Relations between the coordinates of a point in rectangular and cylindrical coordinate systems:

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad \text{and} \qquad z = z$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial T}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{e}_{\text{gen}} = \rho c\frac{\partial T}{\partial t}$$

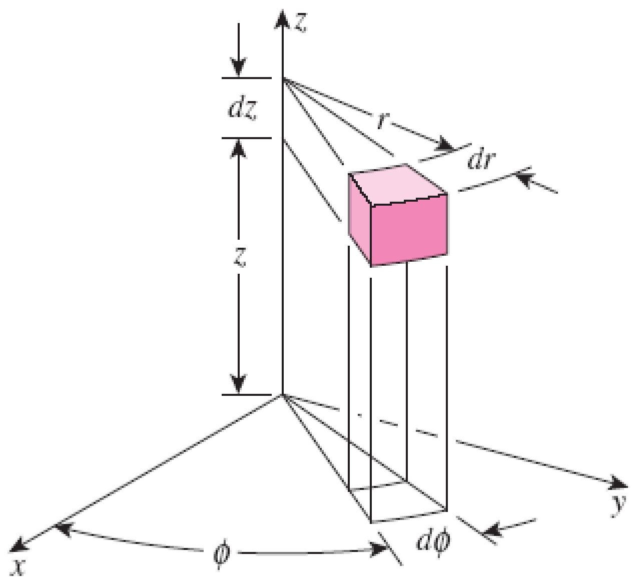


FIGURE 2–22

A differential volume element in cylindrical coordinates.

Spherical Coordinates

Relations between the coordinates of a point in rectangular and spherical coordinate systems:

$$x = r\cos\phi\sin\theta, \qquad y = r\sin\phi\sin\theta, \qquad \text{and} \qquad z = \cos\theta$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(k\frac{\partial T}{\partial\phi}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(k\sin\theta\frac{\partial T}{\partial\theta}\right) + \dot{e}_{\text{gen}} = \rho c\frac{\partial T}{\partial t}$$

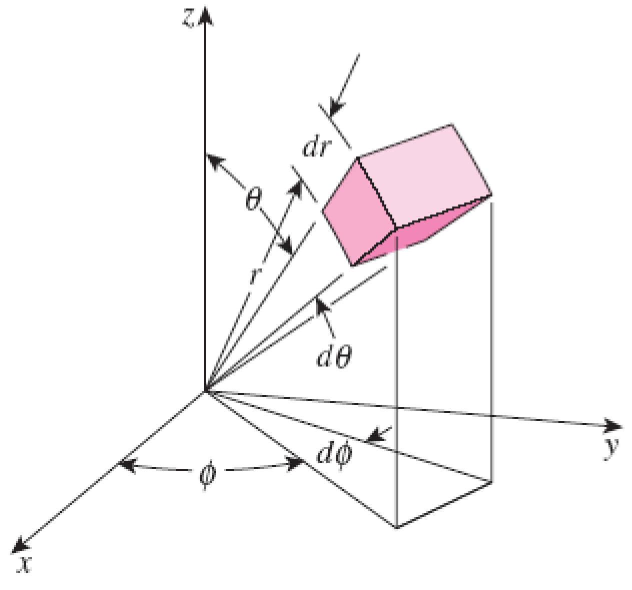


FIGURE 2–23

A differential volume element in spherical coordinates.

EXAMPLE 2–5 Heat Conduction in a Short Cylinder

A short cylindrical metal billet of radius R and height h is heated in an oven to a temperature of 300°C throughout and is then taken out of the oven and allowed to cool in ambient air at $T_\infty = 20°C$ by convection and radiation. Assuming the billet is cooled uniformly from all outer surfaces and the variation of the thermal conductivity of the material with temperature is negligible, obtain the differential equation that describes the variation of the temperature in the billet during this cooling process.

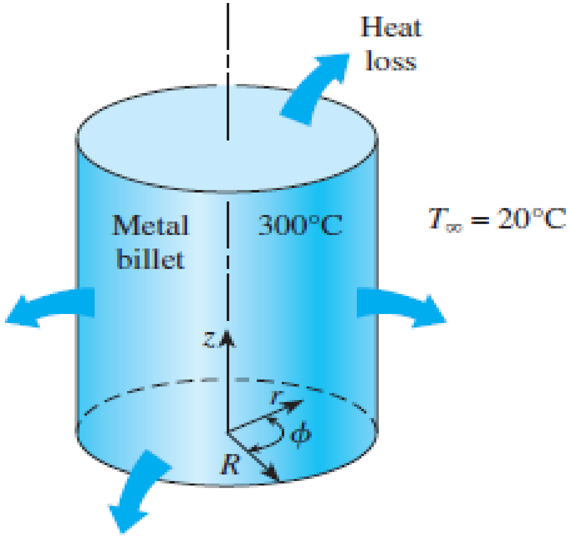


FIGURE 2–24
Schematic for Example 2–5.

SOLUTION A short cylindrical billet is cooled in ambient air. The differential equation for the variation of temperature is to be obtained.

Analysis The billet shown in Fig. 2–24 is initially at a uniform temperature and is cooled uniformly from the top and bottom surfaces in the z-direction as well as the lateral surface in the radial r-direction. Also, the temperature at any point in the ball changes with time during cooling. Therefore, this is a two-dimensional transient heat conduction problem since the temperature within the billet changes with the radial and axial distances r and z and with time t. That is, $T = T(r, z, t)$.

The thermal conductivity is given to be constant, and there is no heat generation in the billet. Therefore, the differential equation that governs the variation of temperature in the billet in this case is obtained from Eq. 2–43 by setting the heat generation term and the derivatives with respect to ϕ equal to zero. We obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) = \rho c\frac{\partial T}{\partial t}$$

In the case of constant thermal conductivity, it reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

which is the desired equation.

Discussion Note that the boundary and initial conditions have no effect on the differential equation.

BOUNDARY AND INITIAL CONDITIONS

The description of a heat transfer problem in a medium is not complete without a full description of the thermal conditions at the bounding surfaces of the medium.

Boundary conditions: The *mathematical expressions* of the thermal conditions at the boundaries.

The temperature at any point on the wall at a specified time depends on the condition of the geometry at the beginning of the heat conduction process.

Such a condition, which is usually specified at time ***t* = 0**, is called the **initial condition**, which is a mathematical expression for the temperature distribution of the medium initially.

$$T(x, y, z, 0) = f(x, y, z)$$

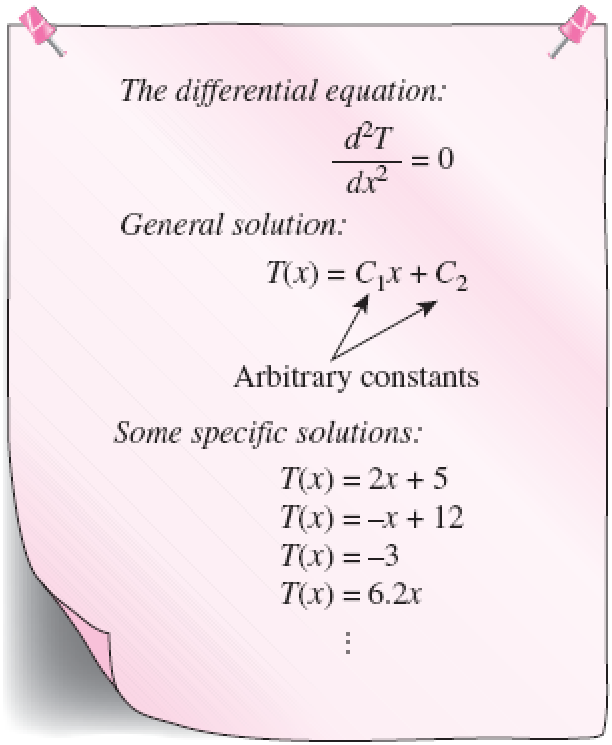


FIGURE 2–25

The general solution of a typical differential equation involves arbitrary constants, and thus an infinite number of solutions.

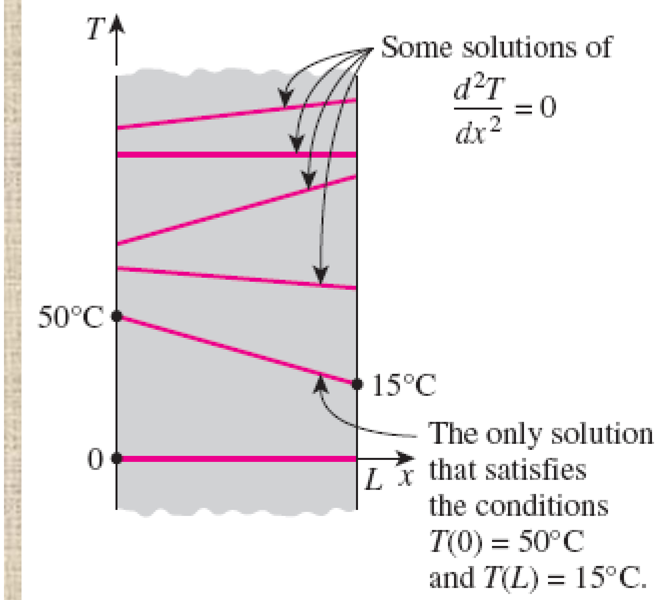


FIGURE 2–26

To describe a heat transfer problem completely, two boundary conditions must be given for each direction along which heat transfer is significant.

Boundary Conditions

- Specified Temperature Boundary Condition
- Specified Heat Flux Boundary Condition
- Convection Boundary Condition
- Radiation Boundary Condition
- Interface Boundary Conditions
- Generalized Boundary Conditions

1 Specified Temperature Boundary Condition

The *temperature* of an exposed surface can usually be measured directly and easily.

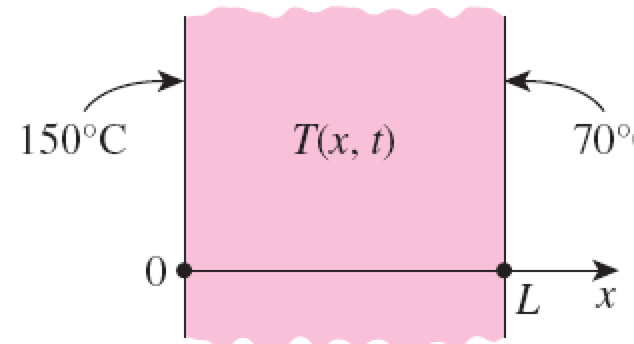
Therefore, one of the easiest ways to specify the thermal conditions on a surface is to specify the temperature.

For one-dimensional heat transfer through a plane wall of thickness L, for example, the specified temperature boundary conditions can be expressed as

$$T(0, t) = T_1$$
$$T(L, t) = T_2$$

where T_1 and T_2 are the specified temperatures at surfaces at $x = 0$ and $x = L$, respectively.

The specified temperatures can be constant, which is the case for steady heat conduction, or may vary with time.



$T(0, t) = 150°C$

$T(L, t) = 70°C$

FIGURE 2–27

Specified temperature boundary conditions on both surfaces of a plane wall.

2 Specified Heat Flux Boundary Condition

The heat flux in the positive *x*-direction anywhere in the medium, including the boundaries, can be expressed by

$$\dot{q} = -k\frac{\partial T}{\partial x} = \left(\begin{array}{c}\text{Heat flux in the}\\ \text{positive } x - \text{direction}\end{array}\right) \qquad (\mathrm{W/m^2})$$

For a plate of thickness *L* subjected to heat flux of 50 W/m² into the medium from both sides, for example, the specified heat flux boundary conditions can be expressed as

$$-k\frac{\partial T(0,t)}{\partial x} = 50 \qquad \text{and} \qquad -k\frac{\partial T(L,t)}{\partial x} = -50$$

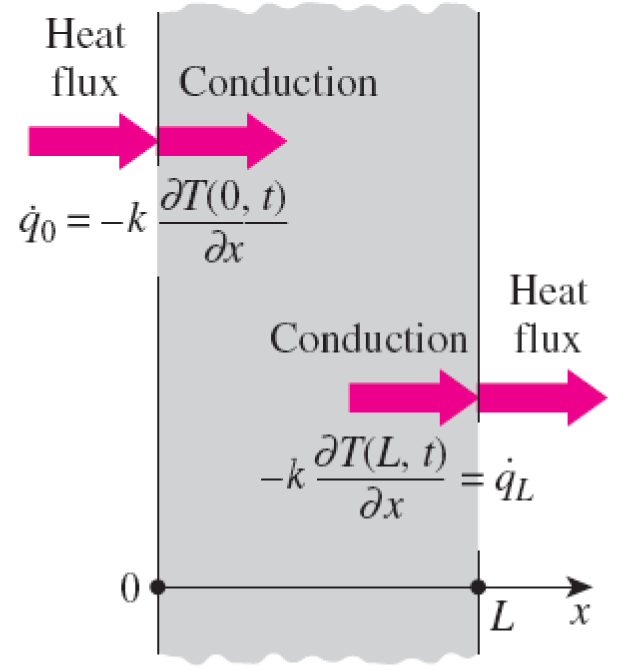


FIGURE 2–28

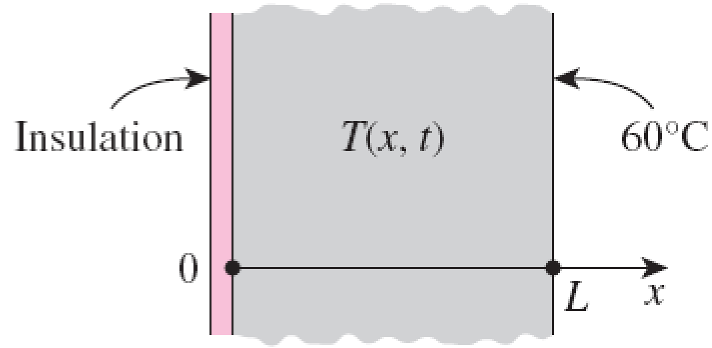
Specified heat flux boundary conditions on both surfaces of a plane wall.

Special Case: Insulated Boundary

A well-insulated surface can be modeled as a surface with a specified heat flux of zero. Then the boundary condition on a perfectly insulated surface (at x = 0, for example) can be expressed as

$$k\frac{\partial T(0, t)}{\partial x} = 0 \qquad \text{or} \qquad \frac{\partial T(0, t)}{\partial x} = 0$$

On an insulated surface, the first derivative of temperature with respect to the space variable (the temperature gradient) in the direction normal to the insulated surface is zero.



$$\frac{\partial T(0, t)}{\partial x} = 0$$

$$T(L, t) = 60°\text{C}$$

FIGURE 2–29

A plane wall with insulation and specified temperature boundary conditions.

Another Special Case: Thermal Symmetry

Some heat transfer problems possess *thermal symmetry* as a result of the symmetry in imposed thermal conditions.

For example, the two surfaces of a large hot plate of thickness *L* suspended vertically in air is subjected to the same thermal conditions, and thus the temperature distribution in one half of the plate is the same as that in the other half.

That is, the heat transfer problem in this plate possesses thermal symmetry about the center plane at $x = L/2$.

Therefore, the center plane can be viewed as an insulated surface, and the thermal condition at this plane of symmetry can be expressed as

$$\frac{\partial T(L/2, t)}{\partial x} = 0$$

which resembles the *insulation* or *zero heat flux* boundary condition.

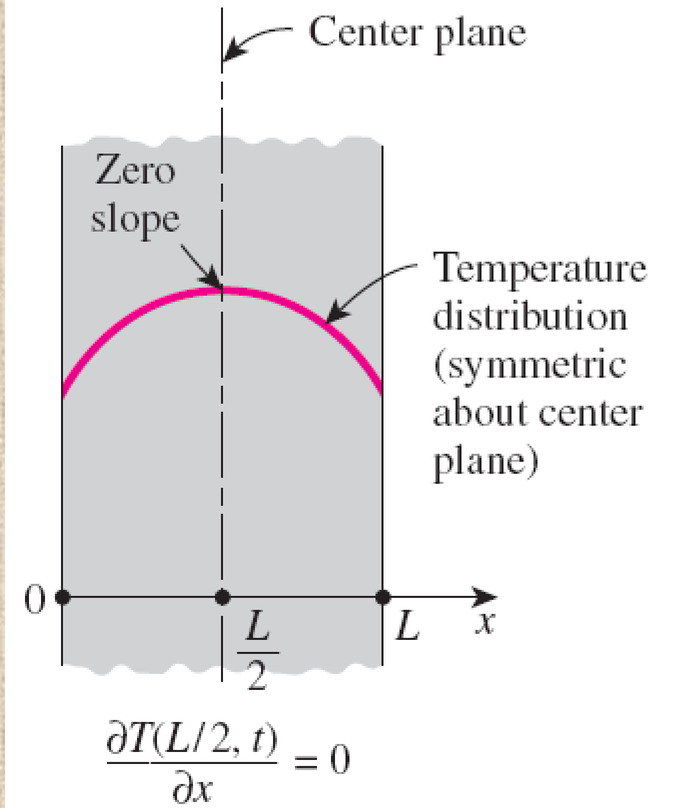


FIGURE 2–30

Thermal symmetry boundary condition at the center plane of a plane wall.

EXAMPLE 2–6 Heat Flux Boundary Condition

Consider an aluminum pan used to cook beef stew on top of an electric range. The bottom section of the pan is $L = 0.3$ cm thick and has a diameter of $D = 20$ cm. The electric heating unit on the range top consumes 800 W of power during cooking, and 90 percent of the heat generated in the heating element is transferred to the pan. During steady operation, the temperature of the inner surface of the pan is measured to be 110°C. Express the boundary conditions for the bottom section of the pan during this cooking process.

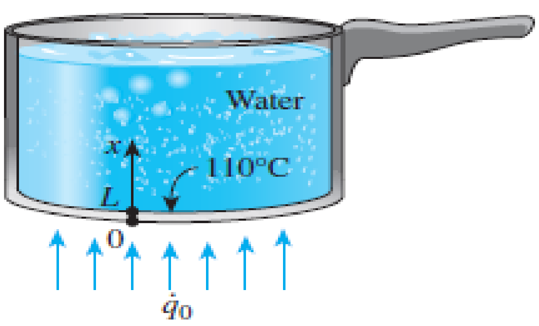


FIGURE 2–31
Schematic for Example 2–6.

SOLUTION An aluminum pan on an electric range top is considered. The boundary conditions for the bottom of the pan are to be obtained.

Analysis The heat transfer through the bottom section of the pan is from the bottom surface toward the top and can reasonably be approximated as being one-dimensional. We take the direction normal to the bottom surfaces of the pan as the x axis with the origin at the outer surface, as shown in Fig. 2–31. Then the inner and outer surfaces of the bottom section of the pan can be represented by $x = 0$ and $x = L$, respectively. During steady operation, the temperature will depend on x only and thus $T = T(x)$.

The boundary condition on the outer surface of the bottom of the pan at $x = 0$ can be approximated as being specified heat flux since it is stated that 90 percent of the 800 W (i.e., 720 W) is transferred to the pan at that surface. Therefore,

$$-k\frac{dT(0)}{dx} = \dot{q}_0$$

where

$$\dot{q}_0 = \frac{\text{Heat transfer rate}}{\text{Bottom surface area}} = \frac{0.720 \text{ kW}}{\pi(0.1 \text{ m})^2} = 22.9 \text{ kW/m}^2$$

The temperature at the inner surface of the bottom of the pan is specified to be 110°C. Then the boundary condition on this surface can be expressed as

$$T(L) = 110°\text{C}$$

where $L = 0.003$ m.

Discussion Note that the determination of the boundary conditions may require some reasoning and approximations.

3 Convection Boundary Condition

For one-dimensional heat transfer in the *x*-direction in a plate of thickness *L*, the convection boundary conditions on both surfaces:

$$\begin{pmatrix}\text{Heat conduction}\\ \text{at the surface in a}\\ \text{selected direction}\end{pmatrix} = \begin{pmatrix}\text{Heat convection}\\ \text{at the surface in}\\ \text{the same direction}\end{pmatrix}$$

$$-k\frac{\partial T(0,t)}{\partial x} = h_1[T_{\infty 1} - T(0,t)]$$

$$-k\frac{\partial T(L,t)}{\partial x} = h_2[T(L,t) - T_{\infty 2}]$$

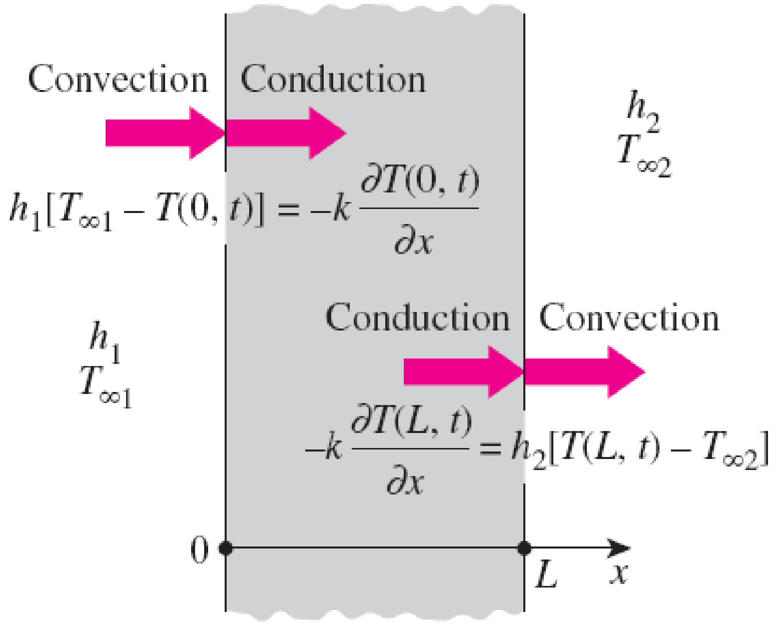


FIGURE 2–32

Convection boundary conditions on the two surfaces of a plane wall.

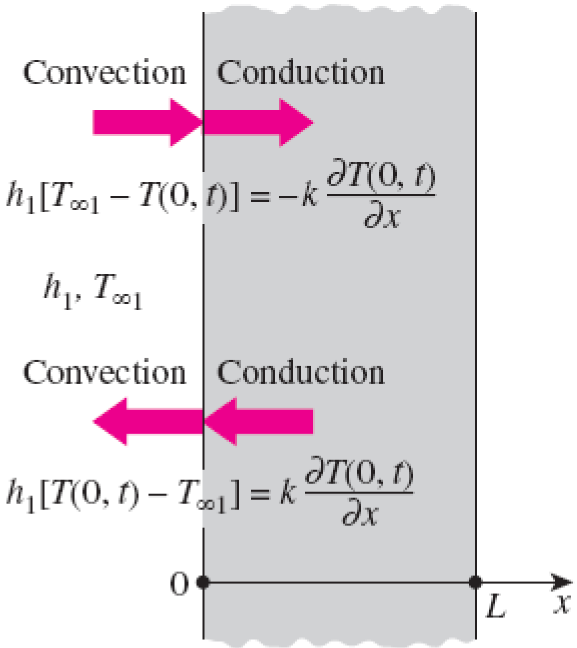


FIGURE 2–33

The assumed direction of heat transfer at a boundary has no effect on the boundary condition expression.

EXAMPLE 2–7 Convection and Insulation Boundary Conditions

Steam flows through a pipe shown in Fig. 2–34 at an average temperature of $T_\infty = 200°C$. The inner and outer radii of the pipe are $r_1 = 8$ cm and $r_2 = 8.5$ cm, respectively, and the outer surface of the pipe is heavily insulated. If the convection heat transfer coefficient on the inner surface of the pipe is $h = 65$ W/m^2·K, express the boundary conditions on the inner and outer surfaces of the pipe during transient periods.

SOLUTION The flow of steam through an insulated pipe is considered. The boundary conditions on the inner and outer surfaces of the pipe are to be obtained.

Analysis During initial transient periods, heat transfer through the pipe material predominantly is in the radial direction, and thus can be approximated as being one-dimensional. Then the temperature within the pipe material changes with the radial distance r and the time t. That is, $T = T(r, t)$.

It is stated that heat transfer between the steam and the pipe at the inner surface is by convection. Then taking the direction of heat transfer to be the positive r direction, the boundary condition on that surface can be expressed as

$$-k\frac{\partial T(r_1, t)}{\partial r} = h[T_\infty - T(r_1)]$$

The pipe is said to be well insulated on the outside, and thus heat loss through the outer surface of the pipe can be assumed to be negligible. Then the boundary condition at the outer surface can be expressed as

$$\frac{\partial T(r_2, t)}{\partial r} = 0$$

Discussion Note that the temperature gradient must be zero on the outer surface of the pipe at all times.

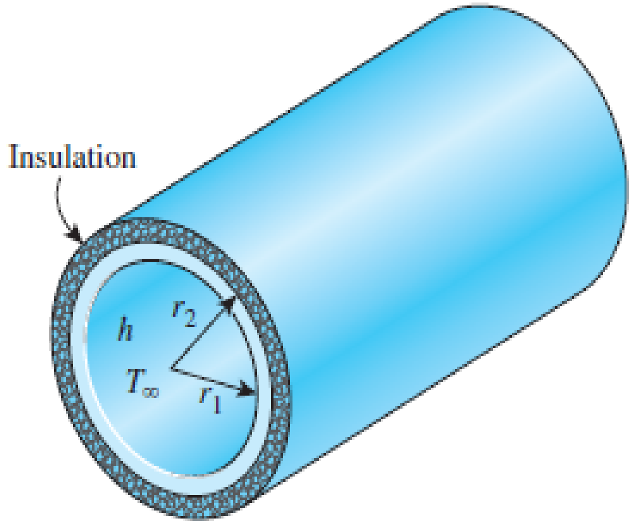


FIGURE 2–34
Schematic for Example 2–7.

4 Radiation Boundary Condition

Radiation boundary condition on a surface:

$$\begin{pmatrix}\text{Heat conduction}\\ \text{at the surface in a}\\ \text{selected direction}\end{pmatrix} = \begin{pmatrix}\text{Radiation exchange}\\ \text{at the surface in}\\ \text{the same direction}\end{pmatrix}$$

For one-dimensional heat transfer in the *x*-direction in a plate of thickness *L*, the radiation boundary conditions on both surfaces can be expressed as

$$-k\frac{\partial T(0,t)}{\partial x} = \varepsilon_1\sigma[T^4_{\text{surr},1} - T(0,t)^4]$$

$$-k\frac{\partial T(L,t)}{\partial x} = \varepsilon_2\sigma[T(L,t)^4 - T^4_{\text{surr},2}]$$

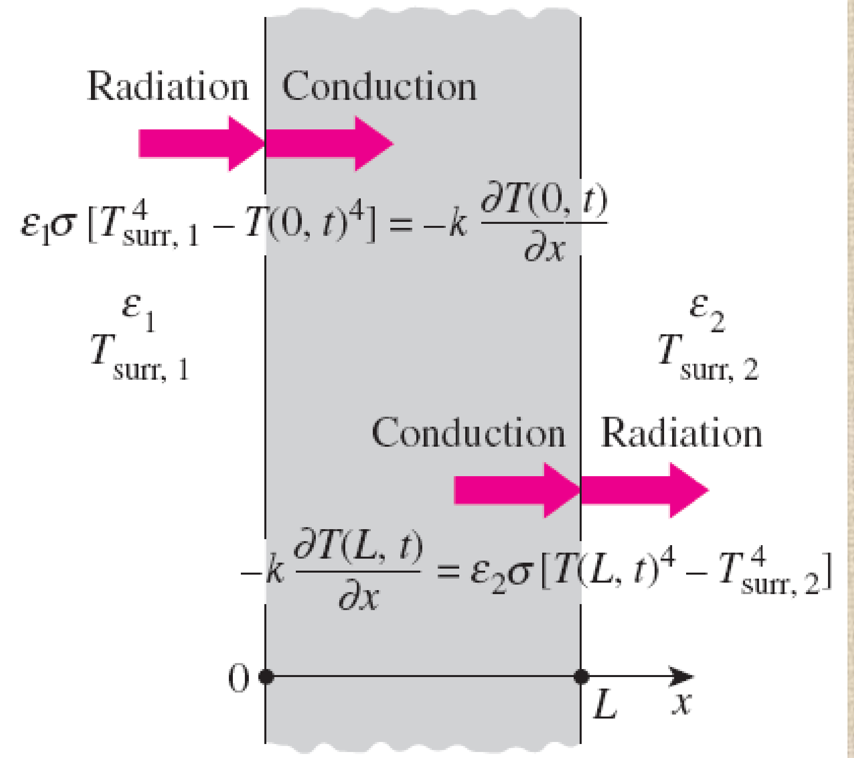


FIGURE 2–35

Radiation boundary conditions on both surfaces of a plane wall.

5 Interface Boundary Conditions

The boundary conditions at an interface are based on the requirements that

(1) two bodies in contact must have the *same temperature* at the area of contact and

(2) an interface (which is a surface) cannot store any energy, and thus the *heat flux* on the two sides of an interface *must be the same.*

The boundary conditions at the interface of two bodies *A* and *B* in perfect contact at $x = x_0$ can be expressed as

$$T_A(x_0, t) = T_B(x_0, t)$$

$$-k_A \frac{\partial T_A(x_0, t)}{\partial x} = -k_B \frac{\partial T_B(x_0, t)}{\partial x}$$

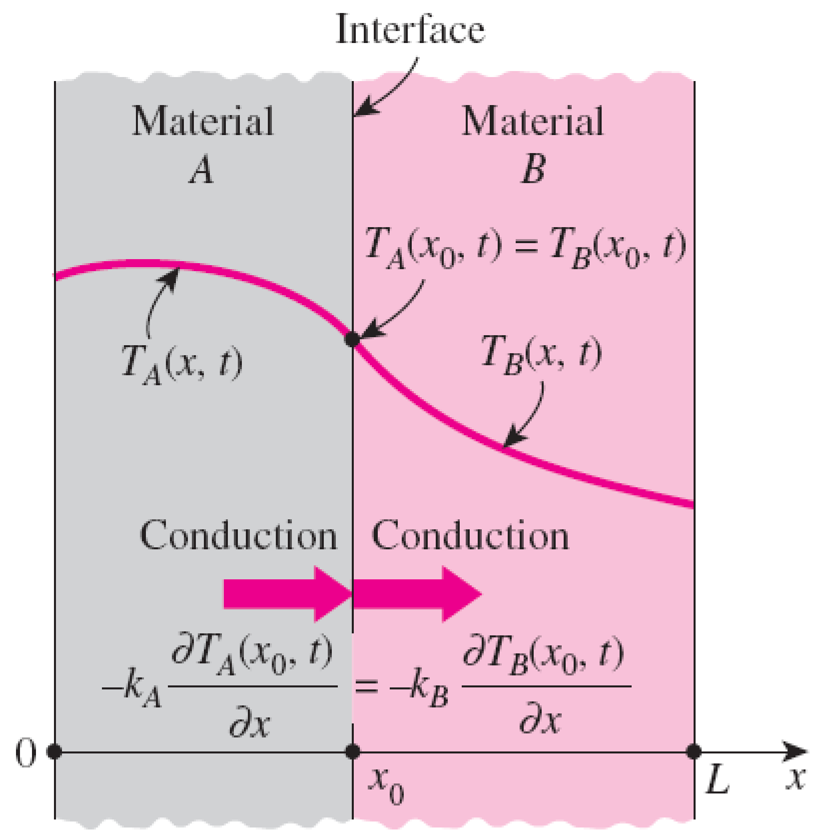


FIGURE 2–36

Boundary conditions at the interface of two bodies in perfect contact.

6 Generalized Boundary Conditions

In general, however, a surface may involve convection, radiation, *and* specified heat flux simultaneously.

The boundary condition in such cases is again obtained from a surface energy balance, expressed as

$$\begin{pmatrix}\text{Heat transfer}\\ \text{to the surface}\\ \text{in all modes}\end{pmatrix} = \begin{pmatrix}\text{Heat transfer}\\ \text{from the surface}\\ \text{in all modes}\end{pmatrix}$$

EXAMPLE 2–8 Combined Convection and Radiation Condition

A spherical metal ball of radius r_o is heated in an oven to a temperature of 300°C throughout and is then taken out of the oven and allowed to cool in ambient air at $T_\infty = 27°C$, as shown in Fig. 2–37. The thermal conductivity of the ball material is $k = 14.4$ W/m·K, and the average convection heat transfer coefficient on the outer surface of the ball is evaluated to be $h = 25$ W/m²·K. The emissivity of the outer surface of the ball is $\varepsilon = 0.6$, and the average temperature of the surrounding surfaces is $T_{surr} = 290$ K. Assuming the ball is cooled uniformly from the entire outer surface, express the initial and boundary conditions for the cooling process of the ball.

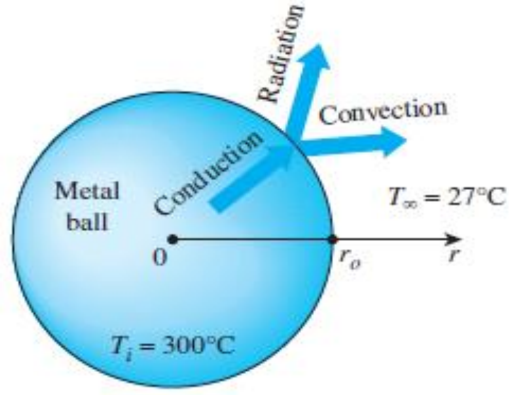


FIGURE 2–37
Schematic for Example 2–8.

SOLUTION The cooling of a hot spherical metal ball is considered. The initial and boundary conditions are to be obtained.

Analysis The ball is initially at a uniform temperature and is cooled uniformly from the entire outer surface. Therefore, this is a one-dimensional transient heat transfer problem since the temperature within the ball changes with the radial distance r and the time t. That is, $T = T(r, t)$. Taking the moment the ball is removed from the oven to be $t = 0$, the initial condition can be expressed as

$$T(r, 0) = T_i = 300°C$$

The problem possesses symmetry about the midpoint ($r = 0$) since the isotherms in this case are concentric spheres, and thus no heat is crossing the midpoint of the ball. Then the boundary condition at the midpoint can be expressed as

$$\frac{\partial T(0, t)}{\partial r} = 0$$

The heat conducted to the outer surface of the ball is lost to the environment by convection and radiation. Then taking the direction of heat transfer to be the positive r direction, the boundary condition on the outer surface can be expressed as

$$-k\frac{\partial T(r_o, t)}{\partial r} = h[T(r_o) - T_\infty] + \varepsilon\sigma[T(r_o)^4 - T_{surr}^4]$$

EXAMPLE 2–9 Combined Convection, Radiation, and Heat Flux

Consider the south wall of a house that is $L = 0.2$ m thick. The outer surface of the wall is exposed to solar radiation and has an absorptivity of $\alpha = 0.5$ for solar energy. The interior of the house is maintained at $T_{\infty 1} = 20°C$, while the ambient air temperature outside remains at $T_{\infty 2} = 5°C$. The sky, the ground, and the surfaces of the surrounding structures at this location can be modeled as a surface at an effective temperature of $T_{sky} = 255$ K for radiation exchange on the outer surface. The radiation exchange between the inner surface of the wall and the surfaces of the walls, floor, and ceiling it faces is negligible. The convection heat transfer coefficients on the inner and the outer surfaces of the wall are $h_1 = 6$ W/m²·K and $h_2 = 25$ W/m²·K, respectively. The thermal conductivity of the wall material is $k = 0.7$ W/m·K, and the emissivity of the outer surface is $\varepsilon_2 = 0.9$. Assuming the heat transfer through the wall to be steady and one-dimensional, express the boundary conditions on the inner and the outer surfaces of the wall.

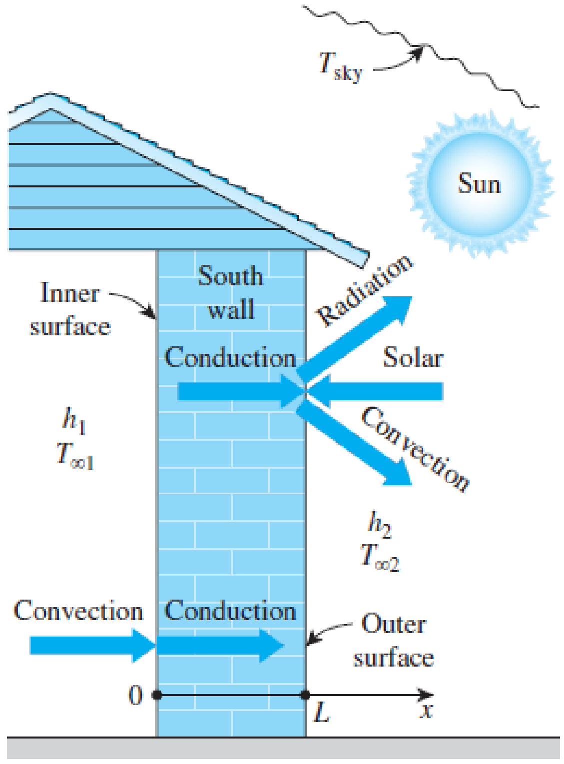


FIGURE 2–38
Schematic for Example 2–9.

SOLUTION The wall of a house subjected to solar radiation is considered. The boundary conditions on the inner and outer surfaces of the wall are to be obtained.

Analysis We take the direction normal to the wall surfaces as the x-axis with the origin at the inner surface of the wall, as shown in Fig. 2–38. The heat transfer through the wall is given to be steady and one-dimensional, and thus the temperature depends on x only and not on time. That is, $T = T(x)$.

The boundary condition on the inner surface of the wall at $x = 0$ is a typical convection condition since it does not involve any radiation or specified heat flux. Taking the direction of heat transfer to be the positive x-direction, the boundary condition on the inner surface can be expressed as

$$-k\frac{dT(0)}{dx} = h_1[T_{\infty 1} - T(0)]$$

The boundary condition on the outer surface at $x = 0$ is quite general as it involves conduction, convection, radiation, and specified heat flux. Again taking the direction of heat transfer to be the positive x-direction, the boundary condition on the outer surface can be expressed as

$$-k\frac{dT(L)}{dx} = h_2[T(L) - T_{\infty 2}] + \varepsilon_2\sigma[T(L)^4 - T_{\text{sky}}^4] - \alpha\dot{q}_{\text{solar}}$$

where $\dot{q}_{\text{solar}}$ is the incident solar heat flux.

Discussion Assuming the opposite direction for heat transfer would give the same result multiplied by -1, which is equivalent to the relation here. All the quantities in these relations are known except the temperatures and their derivatives at the two boundaries.

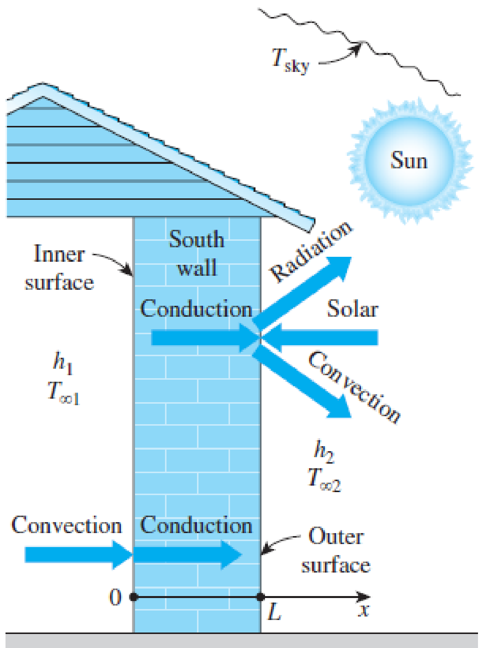


FIGURE 2–38
Schematic for Example 2–9.

SOLUTION OF STEADY ONE-DIMENSIONAL HEAT CONDUCTION PROBLEMS

In this section we will solve a wide range of heat conduction problems in rectangular, cylindrical, and spherical geometries.

We will limit our attention to problems that result in *ordinary differential equations* such as the *steady one-dimensional* heat conduction problems. We will also assume *constant thermal conductivity*.

The solution procedure for solving heat conduction problems can be summarized as

(1) *formulate* the problem by obtaining the applicable differential equation in its simplest form and specifying the boundary conditions,

(2) Obtain the *general solution* of the differential equation, and

(3) apply the *boundary conditions* and determine the arbitrary constants in the general solution.

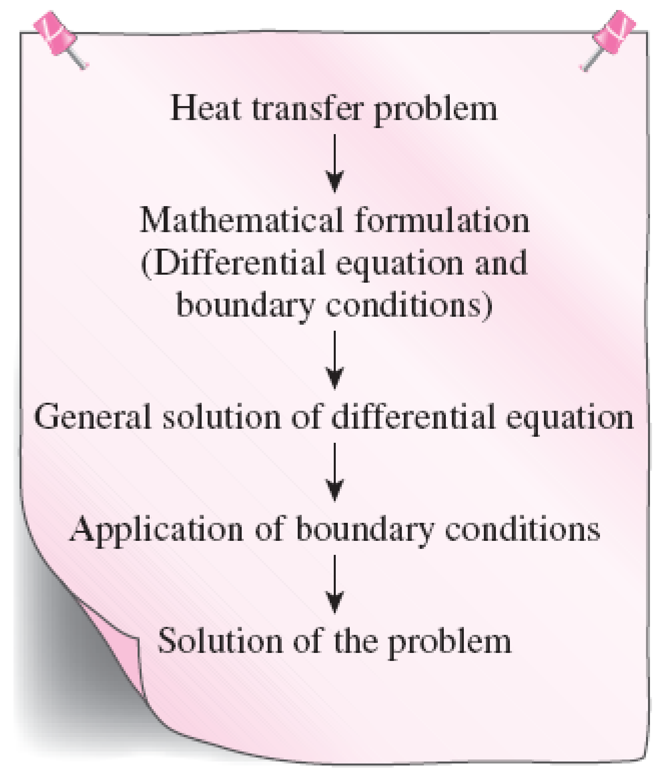


FIGURE 2–39

Basic steps involved in the solution of heat transfer problems.

EXAMPLE 2–10 Heat Conduction in a Plane Wall

Consider a large plane wall of thickness $L = 0.2$ m, thermal conductivity $k = 1.2$ W/m·K, and surface area $A = 15$ m^2. The two sides of the wall are maintained at constant temperatures of $T_1 = 120°C$ and $T_2 = 50°C$, respectively, as shown in Fig. 2–40. Determine (*a*) the variation of temperature within the wall and the value of temperature at $x = 0.1$ m and (*b*) the rate of heat conduction through the wall under steady conditions.

SOLUTION A plane wall with specified surface temperatures is given. The variation of temperature and the rate of heat transfer are to be determined.

Assumptions **1** Heat conduction is steady. **2** Heat conduction is one-dimensional since the wall is large relative to its thickness and the thermal conditions on both sides are uniform. **3** Thermal conductivity is constant. **4** There is no heat generation.

Properties The thermal conductivity is given to be $k = 1.2$ W/m·K.

Analysis (*a*) Taking the direction normal to the surface of the wall to be the *x*-direction, the differential equation for this problem can be expressed as

$$\frac{d^2T}{dx^2} = 0$$

with boundary conditions

$$T(0) = T_1 = 120°C$$
$$T(L) = T_2 = 50°C$$

The differential equation is linear and second order, and a quick inspection of it reveals that it has a single term involving derivatives and no terms involving the unknown function T as a factor. Thus, it can be solved by direct integration. Noting that an integration reduces the order of a derivative by one, the general solution of the differential equation above can be obtained by two simple successive integrations, each of which introduces an integration constant.

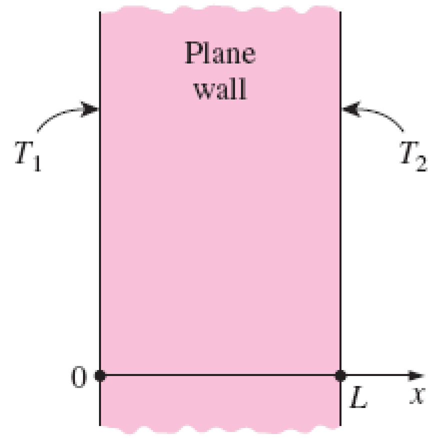


FIGURE 2–40

Schematic for Example 2–10.

Integrating the differential equation once with respect to x yields

$$\frac{dT}{dx} = C_1$$

where C_1 is an arbitrary constant. Notice that the order of the derivative went down by one as a result of integration. As a check, if we take the derivative of this equation, we will obtain the original differential equation. This equation is not the solution yet since it involves a derivative.

Integrating one more time, we obtain

$$T(x) = C_1 x + C_2$$

which is the general solution of the differential equation (Fig. 2–41). The general solution in this case resembles the general formula of a straight line whose slope is C_1 and whose value at $x = 0$ is C_2. This is not surprising since the second derivative represents the change in the slope of a function, and a zero second derivative indicates that the slope of the function remains constant. Therefore, *any straight line* is a solution of this differential equation.

The general solution contains two unknown constants C_1 and C_2, and thus we need two equations to determine them uniquely and obtain the specific solution. These equations are obtained by forcing the general solution to satisfy the specified boundary conditions. The application of each condition yields one equation, and thus we need to specify two conditions to determine the constants C_1 and C_2.

When applying a boundary condition to an equation, *all occurrences of the dependent and independent variables and any derivatives are replaced by the specified values.* Thus the only unknowns in the resulting equations are the arbitrary constants.

The first boundary condition can be interpreted as *in the general solution, replace all the x's by zero and T(x) by T_1.* That is (Fig. 2–42),

$$T(0) = C_1 \times 0 + C_2 \quad \rightarrow \quad C_2 = T_1$$

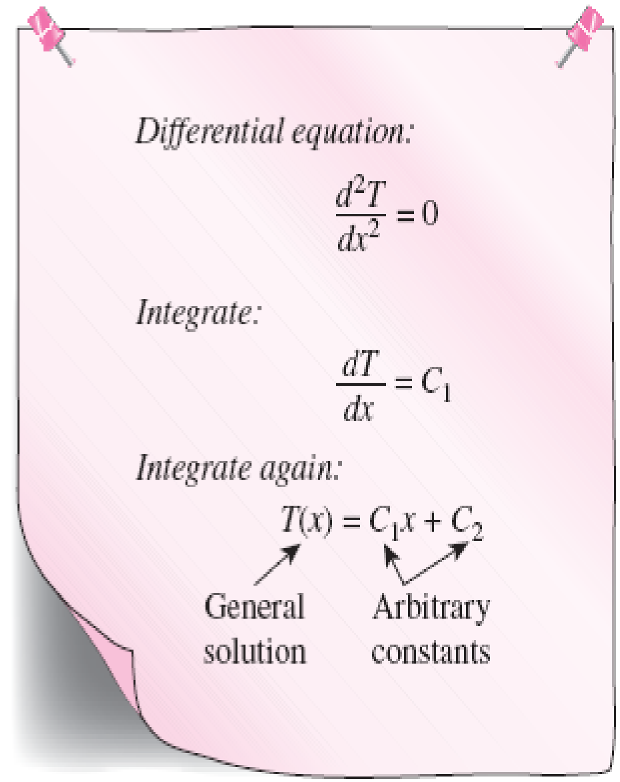


FIGURE 2–41

Obtaining the general solution of a simple second order differential equation by integration.

The second boundary condition can be interpreted as *in the general solution, replace all the x's by L and T(x) by T_2*. That is,

$$T(L) = C_1L + C_2 \quad \rightarrow \quad T_2 = C_1L + T_1 \quad \rightarrow \quad C_1 = \frac{T_2 - T_1}{L}$$

Substituting the C_1 and C_2 expressions into the general solution, we obtain

$$T(x) = \frac{T_2 - T_1}{L}x + T_1 \tag{2–56}$$

which is the desired solution since it satisfies not only the differential equation but also the two specified boundary conditions. That is, differentiating Eq. 2–56 with respect to x twice will give d^2T/dx^2, which is the given differential equation, and substituting $x = 0$ and $x = L$ into Eq. 2–56 gives $T(0) = T_1$ and $T(L) = T_2$, respectively, which are the specified conditions at the boundaries.

Substituting the given information, the value of the temperature at $x = 0.1$ m is determined to be

$$T(0.1 \text{ m}) = \frac{(50 - 120)°\text{C}}{0.2 \text{ m}}(0.1 \text{ m}) + 120°\text{C} = 85°\text{C}$$

(*b*) The rate of heat conduction anywhere in the wall is determined from Fourier's law to be

$$\dot{Q}_{\text{wall}} = -kA\frac{dT}{dx} = -kAC_1 = -kA\frac{T_2 - T_1}{L} = kA\frac{T_1 - T_2}{L} \tag{2–57}$$

The numerical value of the rate of heat conduction through the wall is determined by substituting the given values to be

$$\dot{Q} = kA\frac{T_1 - T_2}{L} = (1.2 \text{ W/m·K})(15 \text{ m}^2)\frac{(120 - 50)°\text{C}}{0.2 \text{ m}} = 6300 \text{ W}$$

Discussion Note that under steady conditions, the rate of heat conduction through a plane wall is constant.

Boundary condition:

$$T(0) = T_1$$

General solution:

$$T(x) = C_1x + C_2$$

Applying the boundary condition:

$$T(\underset{\substack{\uparrow\\ \underbrace{0}_{T_1}}}{x}) = C_1\underset{\substack{\uparrow\\ 0}}{x} + C_2$$

Substituting:

$$T_1 = C_1 \times 0 + C_2 \rightarrow C_2 = T_1$$

It cannot involve x or $T(x)$ after the boundary condition is applied.

FIGURE 2–42

When applying a boundary condition to the general solution at a specified point, all occurrences of the dependent and independent variables should be replaced by their specified values at that point.

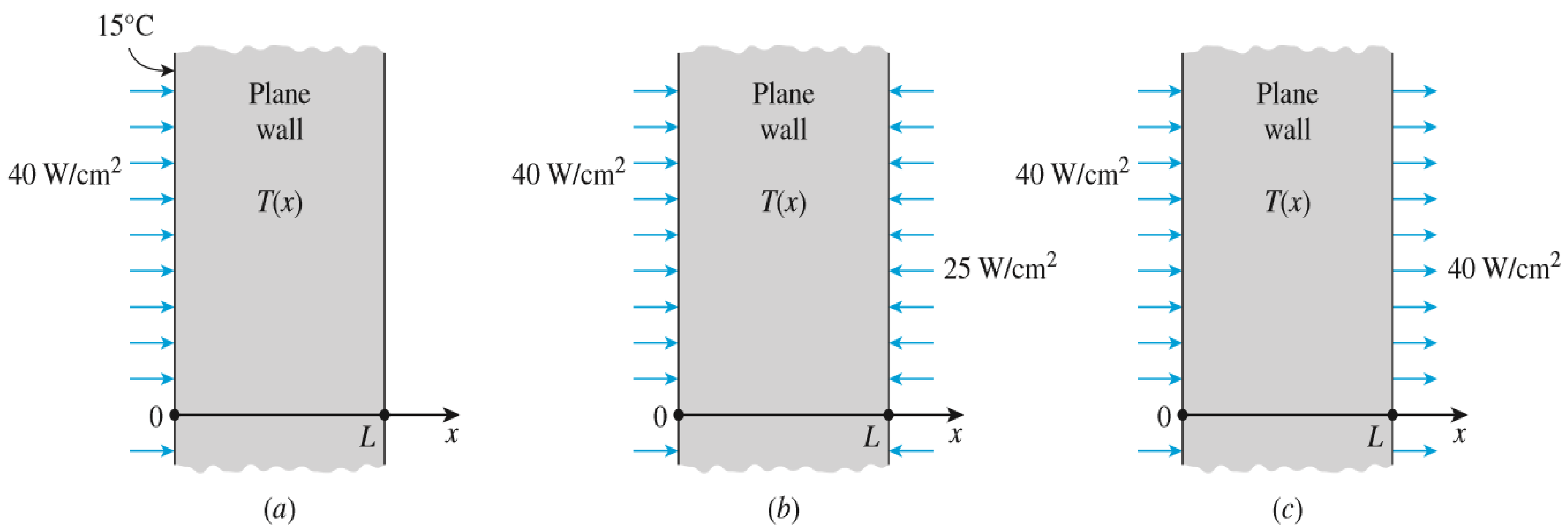
EXAMPLE 2–11 A Wall with Various Sets of Boundary Conditions

Consider steady one-dimensional heat conduction in a large plane wall of thickness L and constant thermal conductivity k with no heat generation. Obtain expressions for the variation of temperature within the wall for the following pairs of boundary conditions (Fig. 2–43):

$$(a)\ -k\frac{dT(0)}{dx} = \dot{q}_0 = 40\ \text{W/cm}^2 \qquad \text{and} \qquad T(0) = T_0 = 15°\text{C}$$

$$(b)\ -k\frac{dT(0)}{dx} = \dot{q}_0 = 40\ \text{W/cm}^2 \qquad \text{and} \qquad -k\frac{dT(L)}{dx} = \dot{q}_L = -25\ \text{W/cm}^2$$

$$(c)\ -k\frac{dT(0)}{dx} = \dot{q}_0 = 40\ \text{W/cm}^2 \qquad \text{and} \qquad -k\frac{dT(L)}{dx} = \dot{q}_L = \dot{q}_0 = 40\ \text{W/cm}^2$$



Analysis This is a steady one-dimensional heat conduction problem with constant thermal conductivity and no heat generation in the medium, and the heat conduction equation in this case can be expressed as (Eq. 2–17)

$$\frac{d^2T}{dx^2} = 0$$

whose general solution was determined in the previous example by direct integration to be

$$T(x) = C_1x + C_2$$

where C_1 and C_2 are two arbitrary integration constants. The specific solutions corresponding to each specified pair of boundary conditions are determined as follows.

(*a*) In this case, both boundary conditions are specified at the same boundary at $x = 0$, and no boundary condition is specified at the other boundary at $x = L$. Noting that

$$\frac{dT}{dx} = C_1$$

the application of the boundary conditions gives

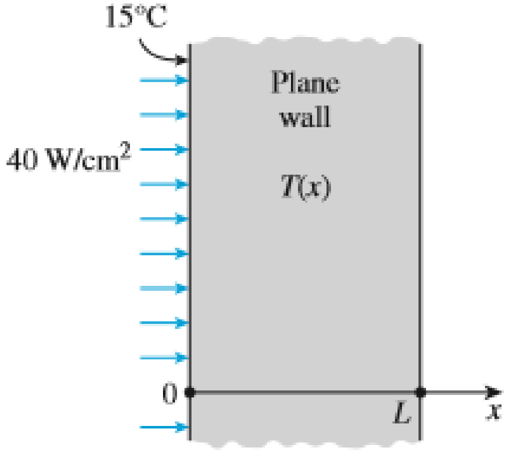
$$-k\frac{dT(0)}{dx} = \dot{q}_0 \quad \rightarrow \quad -kC_1 = \dot{q}_0 \quad \rightarrow \quad C_1 = -\frac{\dot{q}_0}{k}$$

and

$$T(0) = T_0 \quad \rightarrow \quad T_0 = C_1 \times 0 + C_2 \quad \rightarrow \quad C_2 = T_0$$

Substituting, the specific solution in this case is determined to be

$$T(x) = -\frac{\dot{q}_0}{k}x + T_0$$



(*a*)

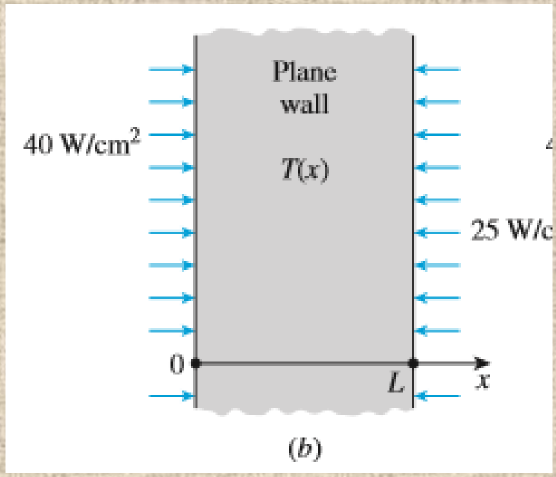
(b) In this case different heat fluxes are specified at the two boundaries. The application of the boundary conditions gives

$$-k\frac{dT(0)}{dx} = \dot{q}_0 \quad \rightarrow \quad -kC_1 = \dot{q}_0 \quad \rightarrow \quad C_1 = -\frac{\dot{q}_0}{k}$$

and

$$-k\frac{dT(L)}{dx} = \dot{q}_L \quad \rightarrow \quad -kC_1 = \dot{q}_L \quad \rightarrow \quad C_1 = -\frac{\dot{q}_L}{k}$$

Since $\dot{q}_0 \neq \dot{q}_L$ and the constant C_1 cannot be equal to two different things at the same time, there is no solution in this case. This is not surprising since this case corresponds to supplying heat to the plane wall from both sides and expecting the temperature of the wall to remain steady (not to change with time). This is impossible.



(b)

(c) In this case, the same values for heat flux are specified at the two boundaries. The application of the boundary conditions gives

$$-k\frac{dT(0)}{dx} = \dot{q}_0 \quad \rightarrow \quad -kC_1 = \dot{q}_0 \quad \rightarrow \quad C_1 = -\frac{\dot{q}_0}{k}$$

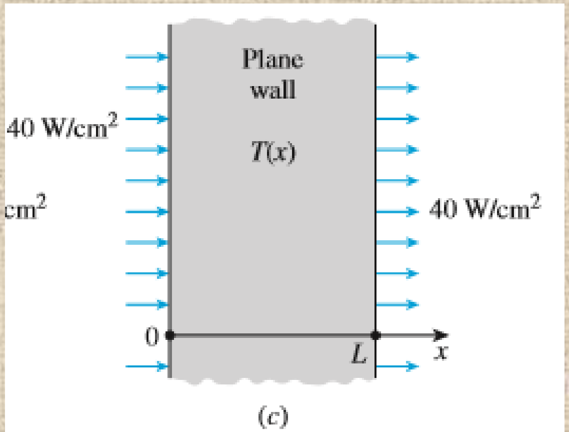
and

$$-k\frac{dT(L)}{dx} = \dot{q}_0 \quad \rightarrow \quad -kC_1 = \dot{q}_0 \quad \rightarrow \quad C_1 = -\frac{\dot{q}_0}{k}$$

Thus, both conditions result in the same value for the constant C_1, but no value for C_2. Substituting, the specific solution in this case is determined to be

$$T(x) = -\frac{\dot{q}_0}{k}x + C_2$$

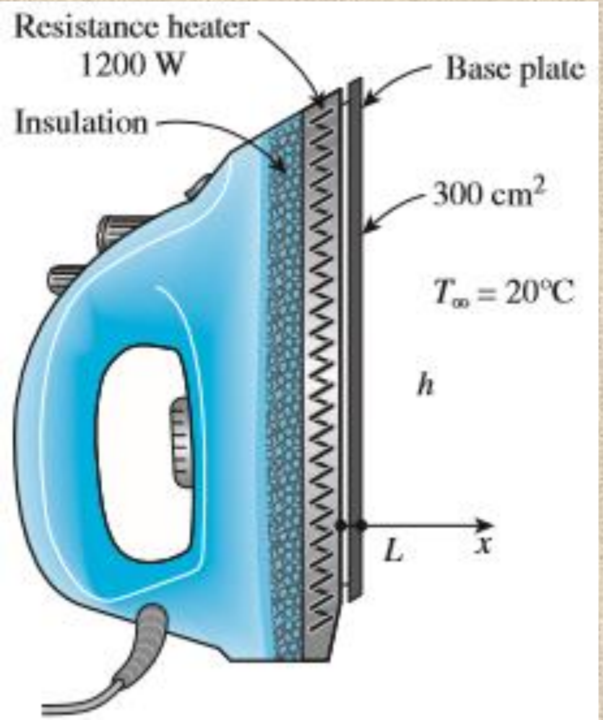
which is not a unique solution since C_2 is arbitrary.



(c)

EXAMPLE 2–12 **Heat Conduction in the Base Plate of an Iron**

Consider the base plate of a 1200-W household iron that has a thickness of L = 0.5 cm, base area of A = 300 cm^2, and thermal conductivity of k = 15 W/m·K. The inner surface of the base plate is subjected to uniform heat flux generated by the resistance heaters inside, and the outer surface loses heat to the surroundings at T_∞ = 20°C by convection, as shown in Fig. 2–45. Taking the convection heat transfer coefficient to be h = 80 W/m^2·K and disregarding heat loss by radiation, obtain an expression for the variation of temperature in the base plate, and evaluate the temperatures at the inner and the outer surfaces.



Analysis The inner surface of the base plate is subjected to uniform heat flux at a rate of

$$\dot{q}_0 = \frac{\dot{Q}_0}{A_{\text{base}}} = \frac{1200\ W}{0.03\ \text{m}^2} = 40{,}000\ \text{W/m}^2$$

The outer side of the plate is subjected to the convection condition. Taking the direction normal to the surface of the wall as the *x*-direction with its origin on the inner surface, the differential equation for this problem can be expressed as (Fig. 2–46)

$$\frac{d^2T}{dx^2} = 0$$

with the boundary conditions

$$-k\frac{dT(0)}{dx} = \dot{q}_0 = 40{,}000\ \text{W/m}^2$$

$$-k\frac{dT(L)}{dx} = h[T(L) - T_\infty]$$

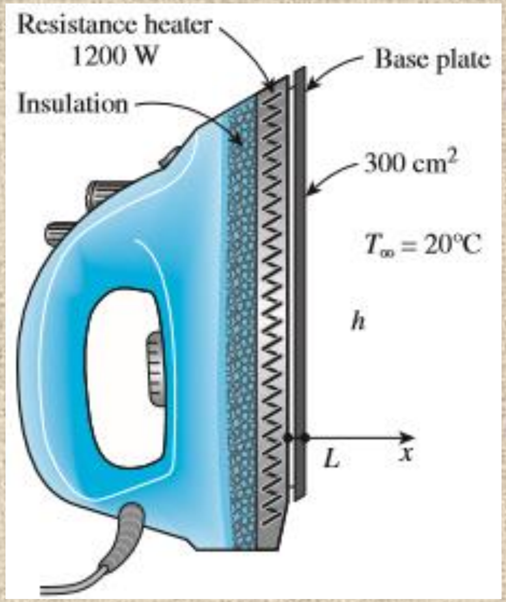
The general solution of the differential equation is again obtained by two successive integrations to be

$$\frac{dT}{dx} = C_1$$

and

$$T(x) = C_1x + C_2 \qquad \textbf{(a)}$$

where C_1 and C_2 are arbitrary constants. Applying the first boundary condition,



$$-k\frac{dT(0)}{dx} = \dot{q}_0 \quad \rightarrow \quad -kC_1 = \dot{q}_0 \quad \rightarrow \quad C_1 = -\frac{\dot{q}_0}{k}$$

Noting that $dT/dx = C_1$ and $T(L) = C_1L + C_2$, the application of the second boundary condition gives

$$-k\frac{dT(L)}{dx} = h[T(L) - T_\infty] \quad \rightarrow \quad -kC_1 = h[(C_1L + C_2) - T_\infty]$$

Substituting $C_1 = -\dot{q}_0/k$ and solving for C_2, we obtain

$$C_2 = T_\infty + \frac{\dot{q}_0}{h} + \frac{\dot{q}_0}{k}L$$

Now substituting C_1 and C_2 into the general solution (*a*) gives

$$T(x) = T_\infty + \dot{q}_0\left(\frac{L-x}{k} + \frac{1}{h}\right) \qquad (b)$$

which is the solution for the variation of the temperature in the plate. The temperatures at the inner and outer surfaces of the plate are determined by substituting $x = 0$ and $x = L$, respectively, into the relation (*b*):

$$T(0) = T_\infty + \dot{q}_0\left(\frac{L}{k} + \frac{1}{h}\right)$$

$$= 20°\text{C} + (40{,}000\ \text{W/m}^2)\left(\frac{0.005\ \text{m}}{15\ \text{W/m·K}} + \frac{1}{80\ \text{W/m}^2\text{·K}}\right) = 533°\text{C}$$

and

$$T(L) = T_\infty + \dot{q}_0\left(0 + \frac{1}{h}\right) = 20°\text{C} + \frac{40{,}000\ \text{W/m}^2}{80\ \text{W/m}^2\text{·K}} = 520°\text{C}$$

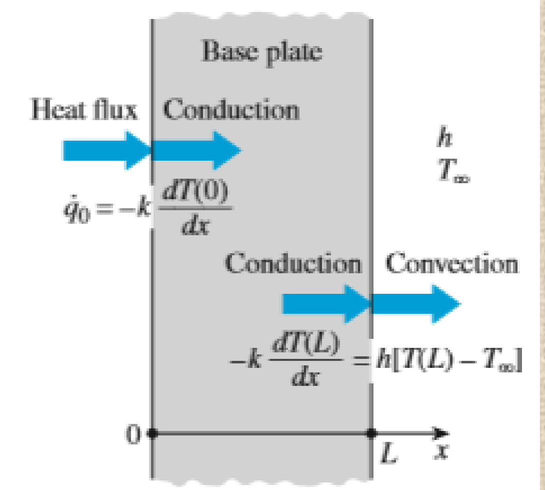
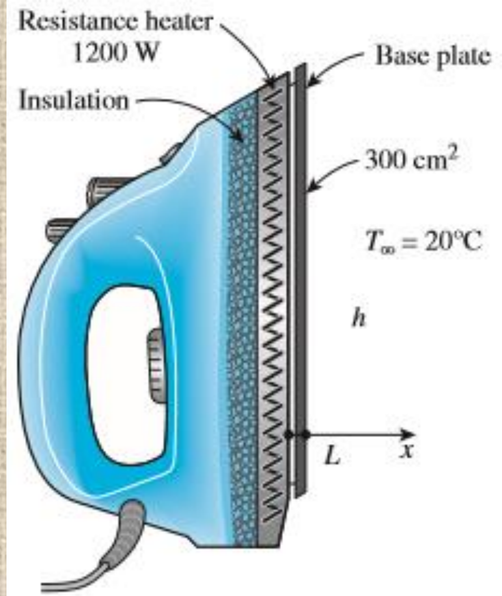


FIGURE 2–46

The boundary conditions on the base plate of the iron discussed in Example 2–12.



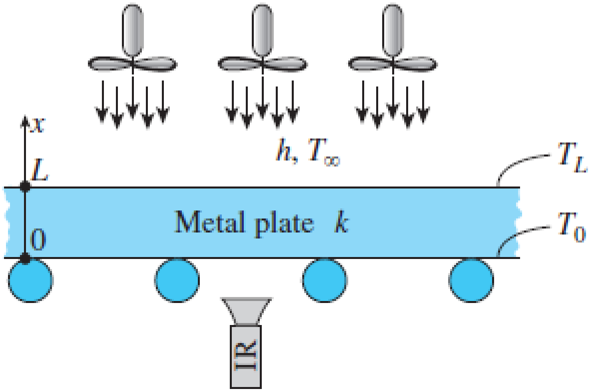


FIGURE 2–47
Schematic for Example 2–13.

PtD **EXAMPLE 2–13 Thermal Burn Prevention in Metal Processing Plant**

In metal processing plants, workers often operate near hot metal surfaces. Exposed hot surfaces are hazards that can potentially cause thermal burns on human skin tissue. Metallic surface with a temperature above 70°C is considered extremely hot. Damage to skin tissue can occur instantaneously upon contact with metallic surface at that temperature. In a plant that processes metal plates, a plate is conveyed through a series of fans to cool its surface in an ambient temperature of 30°C, as shown in Figure 2-47. The plate is 25 mm thick and has a thermal conductivity of 13.5 W/m·K. Temperature at the bottom surface of the plate is monitored by an infrared (IR) thermometer. Obtain an expression for the variation of temperature in the metal plate. The IR thermometer measures the bottom surface of the plate to be 60°C. Determine the minimum value of the convection heat transfer coefficient necessary to keep the top surface below 47°C to avoid instantaneous thermal burn upon accidental contact of hot metal surface with skin tissue.

SOLUTION In this example, the concepts of Prevention through Design (PtD) are applied in conjunction with the solution of steady one-dimensional heat conduction problem. The top surface of the plate is cooled by convection, and temperature at the bottom surface is measured by an IR thermometer. The variation of temperature in the metal plate and the convection heat transfer coefficient necessary to keep the top surface below 47°C are to be determined.

Assumptions **1** Heat conduction is steady and one-dimensional. **2** Thermal conductivity is constant. **3** There is no heat generation in the plate. **4** The bottom surface at $x = 0$ is at constant temperature while the top surface at $x = L$ is subjected to convection.

Properties The thermal conductivity of the metal plate is given to be k = 13.5 W/m·K.

Analysis Taking the direction normal to the surface of the wall to be the x direction with $x = 0$ at the lower surface, the mathematical formulation can be expressed as

$$\frac{d^2T}{dx^2} = 0$$

with boundary conditions

$$T(0) = T_0$$

$$-k\frac{dT(L)}{dx} = h[T(L) - T_\infty]$$

Integrating the differential equation twice with respect to x yields

$$\frac{dT}{dx} = C_1$$

$$T(x) = C_1x + C_2$$

where C_1 and C_2 are arbitrary constants. Applying the first boundary condition yields

$$T(0) = C_1 \times 0 + C_2 = T_0 \rightarrow C_2 = T_0$$

The application of the second boundary condition gives

$$-k\frac{dT(L)}{dx} = h[T(L) - T_\infty] \quad \rightarrow \quad -kC_1 = h(C_1L + C_2 - T_\infty)$$

Solving for C_1 yields

$$C_1 = \frac{h(T_\infty - C_2)}{k + hL} = \frac{T_\infty - T_0}{(k/h) + L}$$

Now substituting C_1 and C_2 into the general solution, the variation of temperature becomes

$$T(x) = \frac{T_\infty - T_0}{(k/h) + L}x + T_0$$

The minimum convection heat transfer coefficient necessary to maintain the top surface below 47°C can be determined from the variation of temperature:

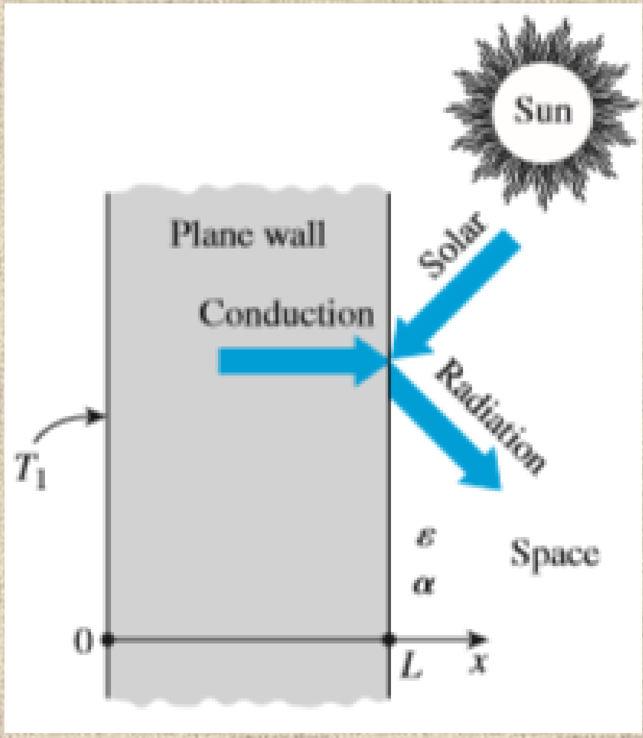
$$T(L) = T_L = \frac{T_\infty - T_0}{(k/h) + L}L + T_0$$

Solving for h gives

$$h = \frac{k}{L}\frac{T_L - T_0}{T_\infty - T_L} = \left(\frac{13.5 \text{ W/m·K}}{0.025 \text{ m}}\right)\frac{(47 - 60)°\text{C}}{(30 - 47)°\text{C}} = \mathbf{413\ W/m^2{\cdot}K}$$

EXAMPLE 2–14 Heat Conduction in a Solar Heated Wall

Consider a large plane wall of thickness $L = 0.06$ m and thermal conductivity $k = 1.2$ W/m·K in space. The wall is covered with white porcelain tiles that have an emissivity of $\varepsilon = 0.85$ and a solar absorptivity of $\alpha = 0.26$, as shown in Fig. 2–48. The inner surface of the wall is maintained at $T_1 = 300$ K at all times, while the outer surface is exposed to solar radiation that is incident at a rate of $\dot{q}_{solar} = 800$ W/m^2. The outer surface is also losing heat by radiation to deep space at 0 K. Determine the temperature of the outer surface of the wall and the rate of heat transfer through the wall when steady operating conditions are reached. What would your response be if no solar radiation was incident on the surface?



Analysis Taking the direction normal to the surface of the wall as the x-direction with its origin on the inner surface, the differential equation for this problem can be expressed as

$$\frac{d^2T}{dx^2} = 0$$

with boundary conditions

$$T(0) = T_1 = 300 \text{ K}$$

$$-k\frac{dT(L)}{dx} = \varepsilon\sigma[T(L)^4 - T_{\text{space}}^4] - \alpha\dot{q}_{\text{solar}}$$

where $T_{\text{space}} = 0$. The general solution of the differential equation is again obtained by two successive integrations to be

$$T(x) = C_1x + C_2 \qquad \textbf{(a)}$$

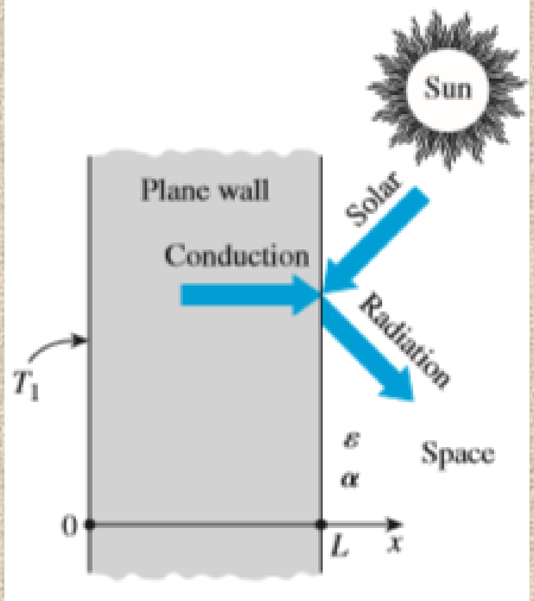
where C_1 and C_2 are arbitrary constants. Applying the first boundary condition yields

$$T(0) = C_1 \times 0 + C_2 \quad \rightarrow \quad C_2 = T_1$$

Noting that $dT/dx = C_1$ and $T(L) = C_1L + C_2 = C_1L + T_1$, the application of the second boundary conditions gives

$$-k\frac{dT(L)}{dx} = \varepsilon\sigma T(L)^4 - \alpha\dot{q}_{\text{solar}} \quad \rightarrow \quad -kC_1 = \varepsilon\sigma(C_1L + T_1)^4 - \alpha\dot{q}_{\text{solar}}$$

Although C_1 is the only unknown in this equation, we cannot get an explicit expression for it because the equation is nonlinear, and thus we cannot get a closed-form expression for the temperature distribution. This should explain why we do our best to avoid nonlinearities in the analysis, such as those associated with radiation.



Let us back up a little and denote the outer surface temperature by $T(L) = T_L$ instead of $T(L) = C_1L + T_1$. The application of the second boundary condition in this case gives

$$-k\frac{dT(L)}{dx} = \varepsilon\sigma T(L)^4 - \alpha\dot{q}_{\text{solar}} \quad \rightarrow \quad -kC_1 = \varepsilon\sigma T_L^4 - \alpha\dot{q}_{\text{solar}}$$

Solving for C_1 gives

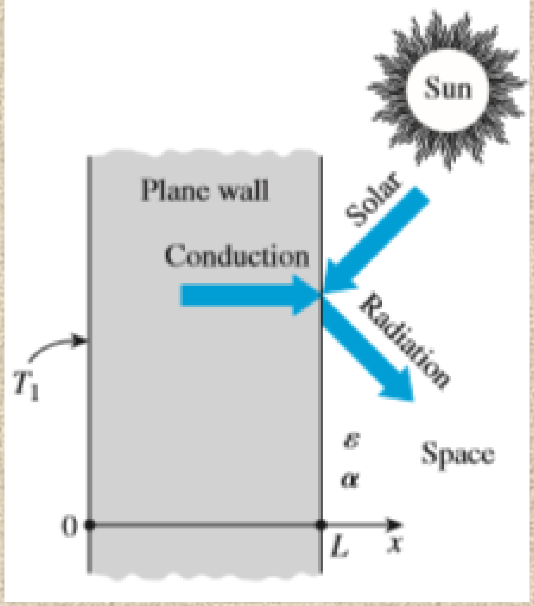
$$C_1 = \frac{\alpha\dot{q}_{\text{solar}} - \varepsilon\sigma T_L^4}{k} \tag{b}$$

Now substituting C_1 and C_2 into the general solution (*a*), we obtain

$$T(x) = \frac{\alpha\dot{q}_{\text{solar}} - \varepsilon\sigma T_L^4}{k}x + T_1 \tag{c}$$

which is the solution for the variation of the temperature in the wall in terms of the unknown outer surface temperature T_L. At $x = L$ it becomes

$$T_L = \frac{\alpha\dot{q}_{\text{solar}} - \varepsilon\sigma T_L^4}{k}L + T_1 \tag{d}$$



which is an implicit relation for the outer surface temperature T_L. Substituting the given values, we get

$$T_L = \frac{0.26 \times (800\ \text{W/m}^2) - 0.85 \times (5.67 \times 10^{-8}\ \text{W/m}^2\cdot\text{K}^4)\, T_L^4}{1.2\ \text{W/m}\cdot\text{K}}(0.06\ \text{m}) + 300\ \text{K}$$

which simplifies to

$$T_L = 310.4 - 0.240975\left(\frac{T_L}{100}\right)^4$$

This equation can be solved by one of the several nonlinear equation solvers available (or by the old fashioned trial-and-error method) to give (Fig. 2–49)

$$T_L = \mathbf{292.7\ K}$$

Knowing the outer surface temperature and knowing that it must remain constant under steady conditions, the temperature distribution in the wall can be determined by substituting the T_L value above into Eq. (*c*):

$$T(x) = \frac{0.26 \times (800\ \text{W/m}^2) - 0.85 \times (5.67 \times 10^{-8}\ \text{W/m}^2\cdot\text{K}^4)(292.7\ \text{K})^4}{1.2\ \text{W/m}\cdot\text{K}}\, x + 300\ \text{K}$$

which simplifies to

$$T(x) = (-121.5\ \text{K/m})x + 300\ \text{K}$$

Note that the outer surface temperature turned out to be lower than the inner surface temperature. Therefore, the heat transfer through the wall is toward the outside despite the absorption of solar radiation by the outer surface. Knowing both the inner and outer surface temperatures of the wall, the steady rate of heat conduction through the wall can be determined from

$$\dot{q} = k\frac{T_1 - T_L}{L} = (1.2\ \text{W/m}\cdot\text{K})\frac{(300 - 292.7)\ \text{K}}{0.06\ \text{m}} = \mathbf{146\ W/m^2}$$

(1) *Rearrange the equation to be solved:*

$$T_L = 310.4 - 0.240975\left(\frac{T_L}{100}\right)^4$$

The equation is in the proper form since the left side consists of T_L only.

(2) *Guess the value of T_L, say 300 K, and substitute into the right side of the equation. It gives*

$$T_L = 290.2\ \text{K}$$

(3) *Now substitute this value of T_L into the right side of the equation and get*

$$T_L = 293.1\ \text{K}$$

(4) *Repeat step (3) until convergence to desired accuracy is achieved. The subsequent iterations give*

$$T_L = 292.6\ \text{K}$$
$$T_L = 292.7\ \text{K}$$
$$T_L = 292.7\ \text{K}$$

Therefore, the solution is $T_L = 292.7$ K. The result is independent of the initial guess.

FIGURE 2–49

A simple method of solving a nonlinear equation is to arrange the equation such that the unknown is alone on the left side while everything else is on the right side, and to iterate after an initial guess until convergence.

EXAMPLE 2–14 Heat Loss through a Steam Pipe

Consider a steam pipe of length $L = 20$ m, inner radius $r_1 = 6$ cm, outer radius $r_2 = 8$ cm, and thermal conductivity $k = 20$ W/m·K, as shown in Fig. 2–49. The inner and outer surfaces of the pipe are maintained at average temperatures of $T_1 = 150°C$ and $T_2 = 60°C$, respectively. Obtain a general relation for the temperature distribution inside the pipe under steady conditions, and determine the rate of heat loss from the steam through the pipe.

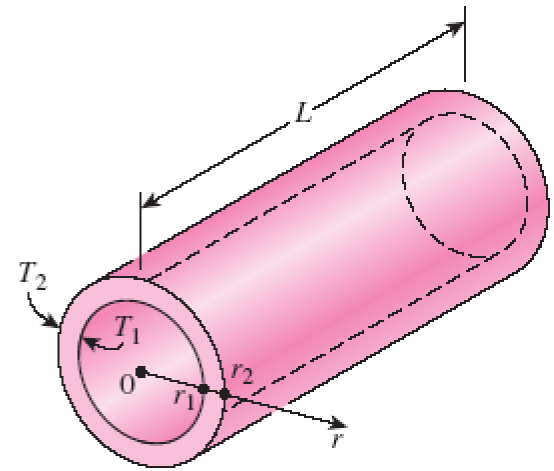


FIGURE 2–49

Schematic for Example 2–14.

SOLUTION A steam pipe is subjected to specified temperatures on its surfaces. The variation of temperature and the rate of heat transfer are to be determined.

Assumptions **1** Heat transfer is steady since there is no change with time. **2** Heat transfer is one-dimensional since there is thermal symmetry about the centerline and no variation in the axial direction, and thus $T = T(r)$. **3** Thermal conductivity is constant. **4** There is no heat generation.

Properties The thermal conductivity is given to be $k = 20$ W/m·K.

Analysis The mathematical formulation of this problem can be expressed as

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

with boundary conditions

$$T(r_1) = T_1 = 150°C$$
$$T(r_2) = T_2 = 60°C$$

Integrating the differential equation once with respect to r gives

$$r\frac{dT}{dr} = C_1$$

where C_1 is an arbitrary constant. We now divide both sides of this equation by r to bring it to a readily integrable form,

$$\frac{dT}{dr} = \frac{C_1}{r}$$

Again integrating with respect to r gives (Fig. 2–50)

$$T(r) = C_1 \ln r + C_2 \qquad \textbf{(}\boldsymbol{a}\textbf{)}$$

We now apply both boundary conditions by replacing all occurrences of r and $T(r)$ in Eq. (a) with the specified values at the boundaries. We get

$$T(r_1) = T_1 \quad \rightarrow \quad C_1 \ln r_1 + C_2 = T_1$$
$$T(r_2) = T_2 \quad \rightarrow \quad C_1 \ln r_2 + C_2 = T_2$$

which are two equations in two unknowns, C_1 and C_2. Solving them simultaneously gives

$$C_1 = \frac{T_2 - T_1}{\ln(r_2/r_1)} \qquad \text{and} \qquad C_2 = T_1 - \frac{T_2 - T_1}{\ln(r_2/r_1)} \ln r_1$$

Substituting them into Eq. (a) and rearranging, the variation of temperature within the pipe is determined to be

$$T(r) = \frac{\ln(r/r_1)}{\ln(r_2/r_1)}(T_2 - T_1) + T_1 \qquad \textbf{(2–58)}$$

The rate of heat loss from the steam is simply the total rate of heat conduction through the pipe, and is determined from Fourier's law to be

$$\dot{Q}_{\text{cylinder}} = -kA\frac{dT}{dr} = -k(2\pi rL)\frac{C_1}{r} = -2\pi kLC_1 = 2\pi kL\frac{T_1 - T_2}{\ln(r_2/r_1)} \qquad \textbf{(2–59)}$$

The numerical value of the rate of heat conduction through the pipe is determined by substituting the given values

$$\dot{Q} = 2\pi(20 \text{ W/m·K})(20 \text{ m})\frac{(150 - 60)°\text{C}}{\ln(0.08/0.06)} = \mathbf{786\ kW}$$

Discussion Note that the total rate of heat transfer through a pipe is constant, but the heat flux $\dot{q} = \dot{Q}/(2\pi rL)$ is not since it decreases in the direction of heat transfer with increasing radius.

Differential equation:

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

Integrate:

$$r\frac{dT}{dr} = C_1$$

Divide by r ($r \neq 0$):

$$\frac{dT}{dr} = \frac{C_1}{r}$$

Integrate again:

$$T(r) = C_1 \ln r + C_2$$

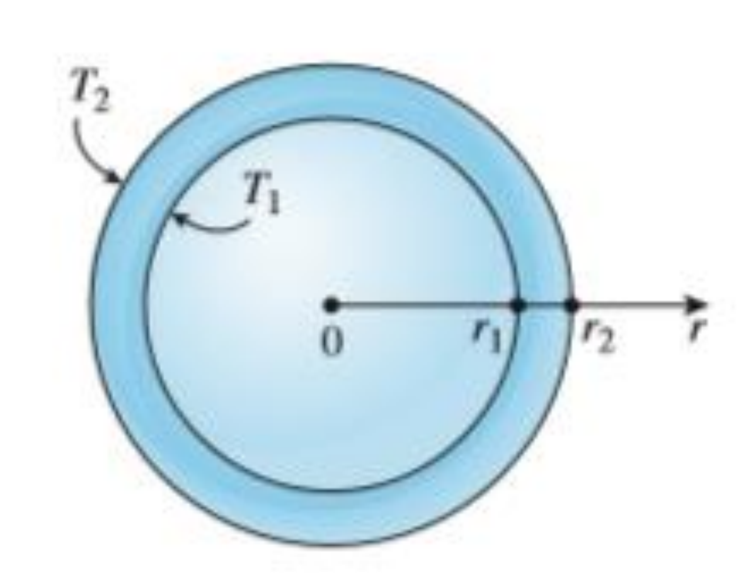
which is the general solution.

FIGURE 2–50

Basic steps involved in the solution of the steady one-dimensional heat conduction equation in cylindrical coordinates.

EXAMPLE 2–16 **Heat Conduction through a Spherical Shell**

Consider a spherical container of inner radius $r_1 = 8$ cm, outer radius $r_2 = 10$ cm, and thermal conductivity $k = 45$ W/m·K, as shown in Fig. 2–52. The inner and outer surfaces of the container are maintained at constant temperatures of $T_1 = 200°C$ and $T_2 = 80°C$, respectively, as a result of some chemical reactions occurring inside. Obtain a general relation for the temperature distribution inside the shell under steady conditions, and determine the rate of heat loss from the container.



Analysis The mathematical formulation of this problem can be expressed as

$$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$$

with boundary conditions

$$T(r_1) = T_1 = 200°C$$
$$T(r_2) = T_2 = 80°C$$

Integrating the differential equation once with respect to r yields

$$r^2\frac{dT}{dr} = C_1$$

where C_1 is an arbitrary constant. We now divide both sides of this equation by r^2 to bring it to a readily integrable form,

$$\frac{dT}{dr} = \frac{C_1}{r^2}$$

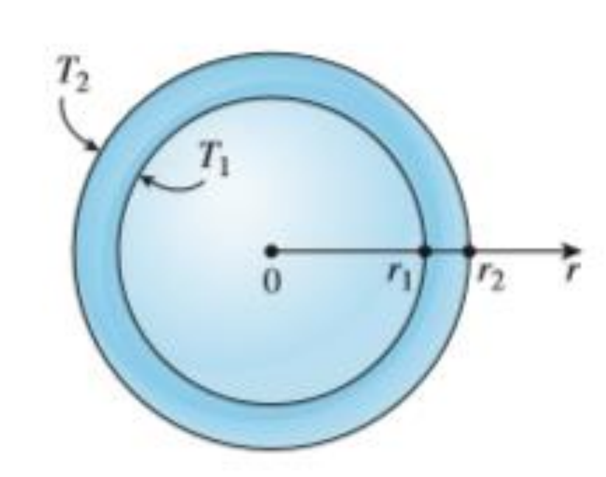
Again integrating with respect to r gives

$$T(r) = -\frac{C_1}{r} + C_2 \qquad (a)$$

We now apply both boundary conditions by replacing all occurrences of r and $T(r)$ in the relation above by the specified values at the boundaries. We get

$$T(r_1) = T_1 \quad \rightarrow \quad -\frac{C_1}{r_1} + C_2 = T_1$$

$$T(r_2) = T_2 \quad \rightarrow \quad -\frac{C_1}{r_2} + C_2 = T_2$$



which are two equations in two unknowns, C_1 and C_2. Solving them simultaneously gives

$$C_1 = -\frac{r_1 r_2}{r_2 - r_1}(T_1 - T_2) \quad \text{and} \quad C_2 = \frac{r_2 T_2 - r_1 T_1}{r_2 - r_1}$$

Substituting into Eq. (*a*), the variation of temperature within the spherical shell is determined to be

$$T(r) = \frac{r_1 r_2}{r(r_2 - r_1)}(T_1 - T_2) + \frac{r_2 T_2 - r_1 T_1}{r_2 - r_1} \qquad \textbf{(2–60)}$$

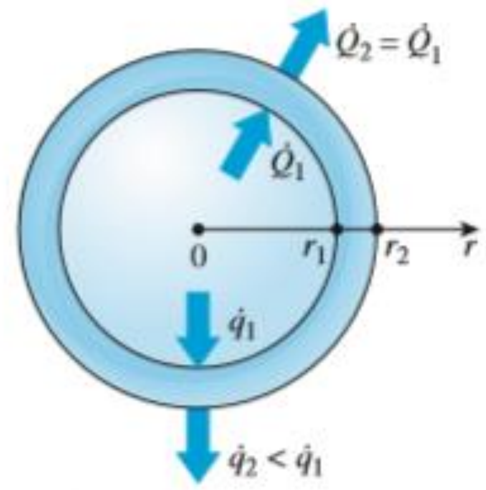
The rate of heat loss from the container is simply the total rate of heat conduction through the container wall and is determined from Fourier's law

$$\dot{Q}_{\text{sphere}} = -kA\frac{dT}{dr} = -k(4\pi r^2)\frac{C_1}{r^2} = -4\pi k C_1 = 4\pi k r_1 r_2 \frac{T_1 - T_2}{r_2 - r_1} \qquad \textbf{(2–61)}$$

The numerical value of the rate of heat conduction through the wall is determined by substituting the given values to be

$$\dot{Q} = 4\pi(45\text{ W/m·K})(0.08\text{ m})(0.10\text{ m})\frac{(200 - 80)°\text{C}}{(0.10 - 0.08)\text{ m}} = \mathbf{27.1\ kW}$$

Discussion Note that the total rate of heat transfer through a spherical shell is constant, but the heat flux $\dot{q} = \dot{Q}/4\pi r^2$ is not since it decreases in the direction of heat transfer with increasing radius as shown in Fig. 2–53.



$$\dot{q}_1 = \frac{\dot{Q}_1}{A_1} = \frac{27.1\text{ kW}}{4\pi(0.08\text{ m})^2} = 337\text{ kW/m}^2$$

$$\dot{q}_2 = \frac{\dot{Q}_2}{A_2} = \frac{27.1\text{ kW}}{4\pi(0.10\text{ m})^2} = 216\text{ kW/m}^2$$

FIGURE 2–53
During steady one-dimensional heat conduction in a spherical (or cylindrical) container, the total rate of heat transfer remains constant, but the heat flux decreases with increasing radius.

HEAT GENERATION IN A SOLID

Many practical heat transfer applications involve the conversion of some form of energy into *thermal* energy in the medium.

Such mediums are said to involve internal *heat generation*, which manifests itself as a rise in temperature throughout the medium.

Some examples of heat generation are

- *resistance heating* in wires,

- exothermic *chemical reactions* in a solid, and

- *nuclear reactions* in nuclear fuel rods

where electrical, chemical, and nuclear energies are converted to heat, respectively.

Heat generation in an electrical wire of outer radius r_o and length L can be expressed as

$$\dot{e}_{\text{gen}} = \frac{\dot{E}_{\text{gen, electric}}}{V_{\text{wire}}} = \frac{I^2 R_e}{\pi r_o^2 L} \qquad (\text{W/m}^3)$$

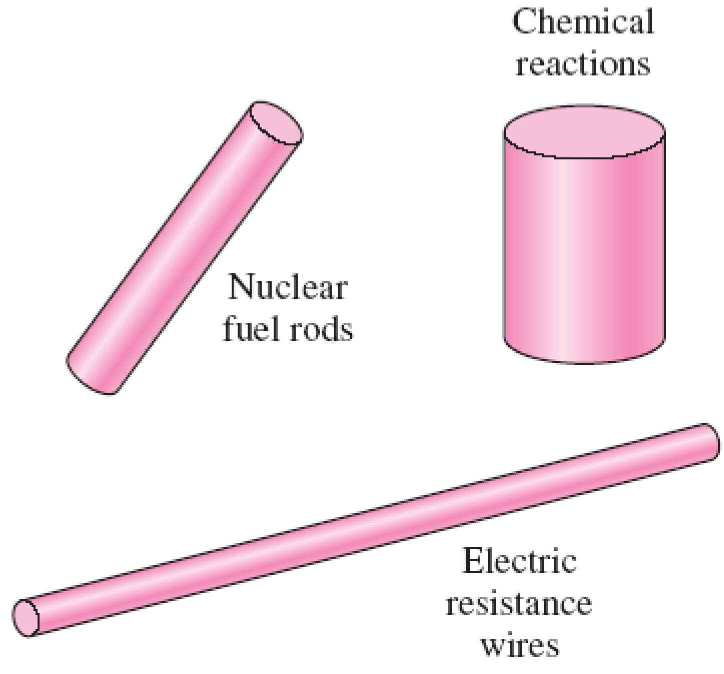


FIGURE 2–53

Heat generation in solids is commonly encountered in practice.

The quantities of major interest in a medium with heat generation are the surface temperature T_s and the maximum temperature T_{max} that occurs in the medium in *steady* operation.

$$\begin{pmatrix}\text{Rate of}\\ \textit{heat transfer}\\ \text{from the solid}\end{pmatrix} = \begin{pmatrix}\text{Rate of}\\ \textit{energy generation}\\ \text{within the solid}\end{pmatrix}$$

$$\dot{Q} = \dot{e}_{gen}V \qquad \text{(W)}$$

$$\dot{Q} = hA_s(T_s - T_\infty) \qquad \text{(W)}$$

$$T_s = T_\infty + \frac{\dot{e}_{gen}V}{hA_s}$$

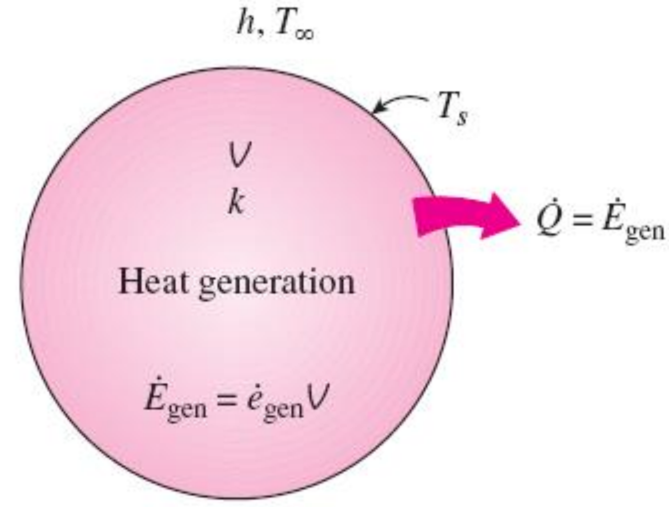


FIGURE 2–54

At steady conditions, the entire heat generated in a solid must leave the solid through its outer surface.

For a large *plane wall* of thickness $2L$ ($A_s = 2A_{wall}$ and $V = 2LA_{wall}$) with both sides of the wall maintained at the same temperature T_s, a long solid *cylinder* of radius r_o ($A_s = 2\pi r_o L$ and $V = \pi r_o^2 L$), and a solid *sphere* of radius r_o ($A_s = 4\pi r_o^2$ and $V = \frac{4}{3}\pi r_o^3$), Eq. 2–66 reduces to

$$T_{s,\,\text{plane wall}} = T_\infty + \frac{\dot{e}_{gen}L}{h}$$

$$T_{s,\,\text{cylinder}} = T_\infty + \frac{\dot{e}_{gen}r_o}{2h}$$

$$T_{s,\,\text{sphere}} = T_\infty + \frac{\dot{e}_{gen}r_o}{3h}$$

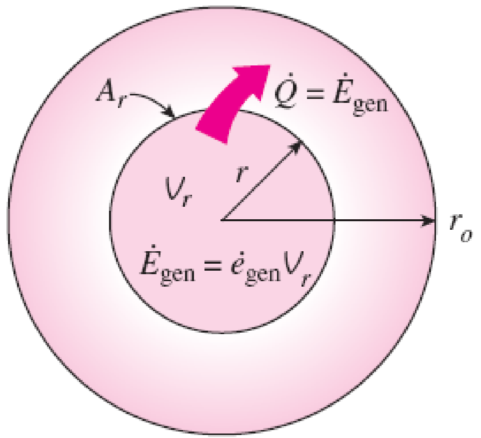


FIGURE 2–55

Heat conducted through a cylindrical shell of radius r is equal to the heat generated within a shell.

Fourier's Law of heat conduction at *r = r*:

$$-kA_r \frac{dT}{dr} = \dot{e}_{gen} V_r$$

where

$$A_r = 2\pi rL \qquad V_r = \pi r^2 L$$

Substituting,

$$-k(2\pi rL)\frac{dT}{dr} = \dot{e}_{gen}(\pi r^2 L) \quad \rightarrow \quad dT = -\frac{\dot{e}_{gen}}{2k} rdr$$

Integrating from $r = 0$ where $T(0) = T_0$ to $r = r_o$ where $T(r_o) = T_s$ yields

$$\Delta T_{\text{max, cylinder}} = T_0 - T_s = \frac{\dot{e}_{gen} r_o^2}{4k}$$

$$\Delta T_{\text{max, sphere}} = \frac{\dot{e}_{gen} r_o^2}{6k}$$

$$\Delta T_{\text{max, plane wall}} = \frac{\dot{e}_{gen} L^2}{2k}$$

Maximum temperature:

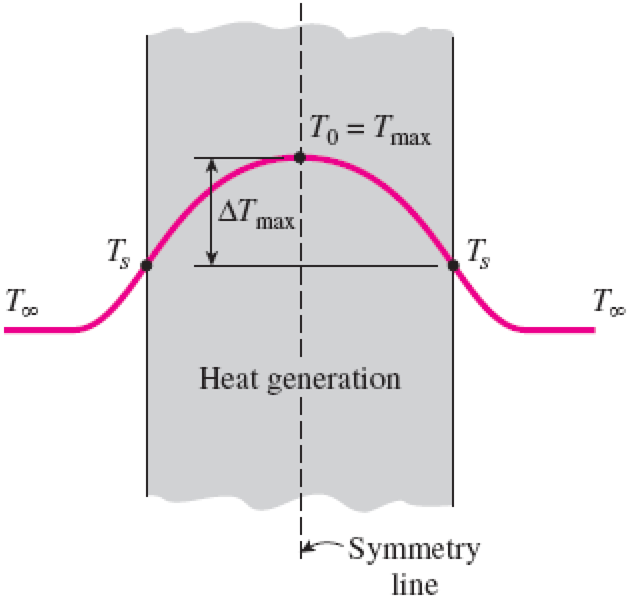


FIGURE 2–56

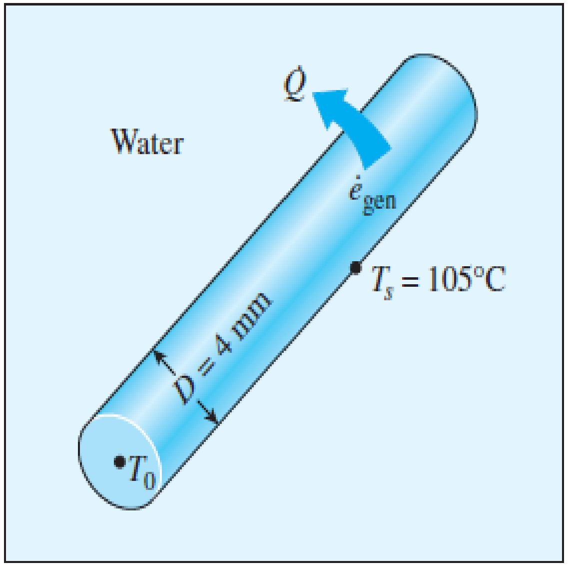
The maximum temperature in a symmetrical solid with uniform heat generation occurs at its center.

$$\Delta T_{\text{max, plane wall}} = \frac{\dot{e}_{\text{gen}} L^2}{2k}$$

$$T_{\text{center}} = T_0 = T_s + \Delta T_{\text{max}}$$

EXAMPLE 2–17 Centerline Temperature of a Resistance Heater

A 2-kW resistance heater wire whose thermal conductivity is $k = 15$ W/m·K has a diameter of $D = 4$ mm and a length of $L = 0.5$ m, and is used to boil water (Fig. 2–58). If the outer surface temperature of the resistance wire is $T_s = 105°C$, determine the temperature at the center of the wire.



SOLUTION The center temperature of a resistance heater submerged in water is to be determined.

Assumptions **1** Heat transfer is steady since there is no change with time. **2** Heat transfer is one-dimensional since there is thermal symmetry about the centerline and no change in the axial direction. **3** Thermal conductivity is constant. **4** Heat generation in the heater is uniform.

Properties The thermal conductivity is given to be $k = 15$ W/m·K.

Analysis The 2-kW resistance heater converts electric energy into heat at a rate of 2 kW. The heat generation per unit volume of the wire is

$$\dot{e}_{gen} = \frac{\dot{E}_{gen}}{V_{wire}} = \frac{\dot{E}_{gen}}{\pi r_o^2 L} = \frac{2000 \text{ W}}{\pi(0.002 \text{ m})^2(0.5 \text{ m})} = 0.318 \times 10^9 \text{ W/m}^3$$

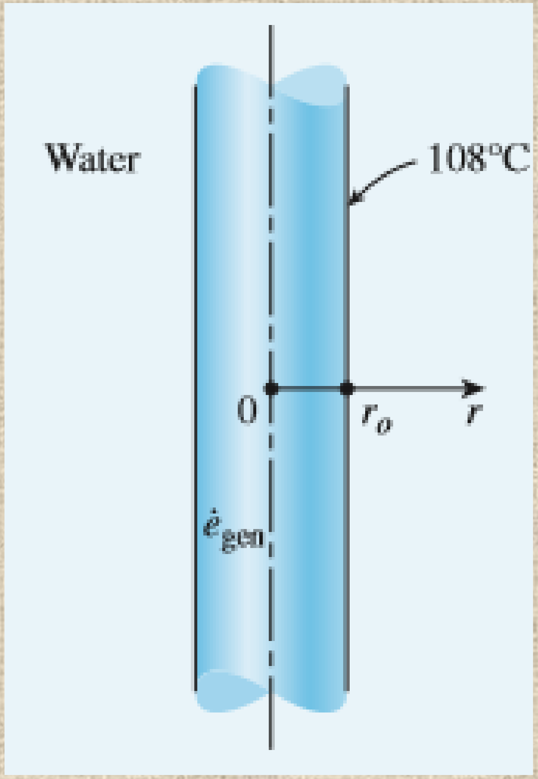
Then the center temperature of the wire is determined from Eq. 2–71 to be

$$T_0 = T_s + \frac{\dot{e}_{gen} r_o^2}{4k} = 105°C + \frac{(0.318 \times 10^9 \text{ W/m}^3)(0.002 \text{ m})^2}{4 \times (15 \text{ W/m·°C})} = 126°C$$

Discussion Note that the temperature difference between the center and the surface of the wire is 21°C. Also, the thermal conductivity units W/m·°C and W/m·K are equivalent.

EXAMPLE 2–18 Variation of Temperature in a Resistance Heater

A long homogeneous resistance wire of radius $r_o = 0.5$ cm and thermal conductivity $k = 13.5$ W/m·°C is being used to boil water at atmospheric pressure by the passage of electric current, as shown in Fig. 2–59. Heat is generated in the wire uniformly as a result of resistance heating at a rate of $\dot{e}_{gen} = 4.3 \times 10^7$ W/m^3. If the outer surface temperature of the wire is measured to be $T_s = 108$°C, obtain a relation for the temperature distribution, and determine the temperature at the centerline of the wire when steady operating conditions are reached.



Analysis The differential equation which governs the variation of temperature in the wire is simply Eq. 2–27,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{e}_{\text{gen}}}{k}=0$$

This is a second-order linear ordinary differential equation, and thus its general solution contains two arbitrary constants. The determination of these constants requires the specification of two boundary conditions, which can be taken to be

$$T(r_o)=T_s=108°\text{C}$$

and

$$\frac{dT(0)}{dr}=0$$

The first boundary condition simply states that the temperature of the outer surface of the wire is 108°C. The second boundary condition is the symmetry condition at the centerline, and states that the maximum temperature in the wire occurs at the centerline, and thus the slope of the temperature at $r = 0$ must be zero (Fig. 2–60). This completes the mathematical formulation of the problem.

Although not immediately obvious, the differential equation is in a form that can be solved by direct integration. Multiplying both sides of the equation by r and rearranging, we obtain

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right)=-\frac{\dot{e}_{\text{gen}}}{k}r$$

Integrating with respect to r gives

$$r\frac{dT}{dr}=-\frac{\dot{e}_{\text{gen}}}{k}\frac{r^2}{2}+C_1 \qquad (a)$$

since the heat generation is constant, and the integral of a derivative of a function is the function itself. That is, integration removes a derivative. It is conve-

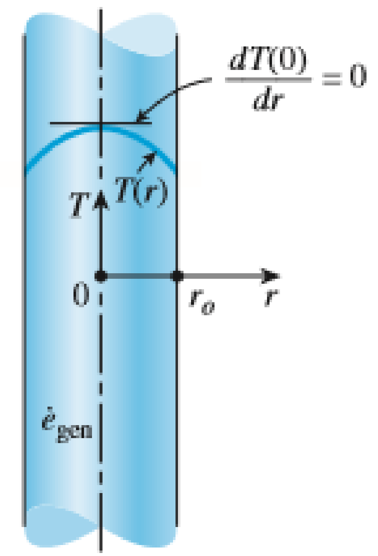


FIGURE 2–60
The thermal symmetry condition at the centerline of a wire in which heat is generated uniformly.

tion is the function itself. That is, integration removes a derivative. It is convenient at this point to apply the second boundary condition, since it is related to the first derivative of the temperature, by replacing all occurrences of r and dT/dr in Eq. (a) by zero. It yields

$$0 \times \frac{dT(0)}{dr} = -\frac{\dot{e}_{gen}}{2k} \times 0 + C_1 \quad \rightarrow \quad C_1 = 0$$

Thus C_1 cancels from the solution. We now divide Eq. (a) by r to bring it to a readily integrable form,

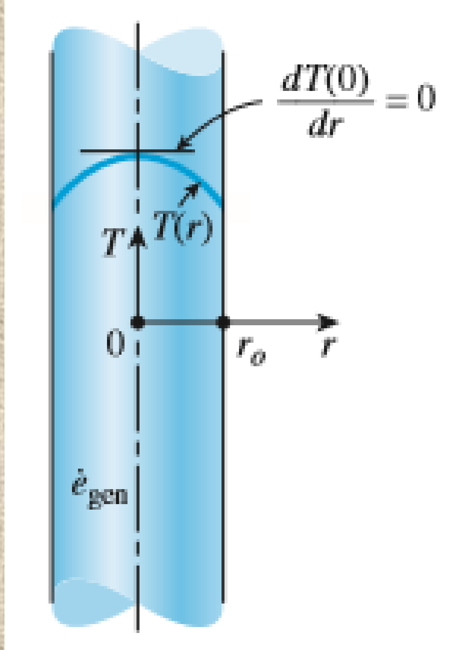
$$\frac{dT}{dr} = -\frac{\dot{e}_{gen}}{2k} r$$

Again integrating with respect to r gives

$$T(r) = -\frac{\dot{e}_{gen}}{4k} r^2 + C_2 \qquad (b)$$

We now apply the first boundary condition by replacing all occurrences of r by r_0 and all occurrences of T by T_s. We get

$$T_s = -\frac{\dot{e}_{gen}}{4k} r_o^2 + C_2 \quad \rightarrow \quad C_2 = T_s + \frac{\dot{e}_{gen}}{4k} r_o^2$$



Substituting this C_2 relation into Eq. (b) and rearranging give

$$T(r) = T_s + \frac{\dot{e}_{gen}}{4k}(r_o^2 - r^2) \qquad (c)$$

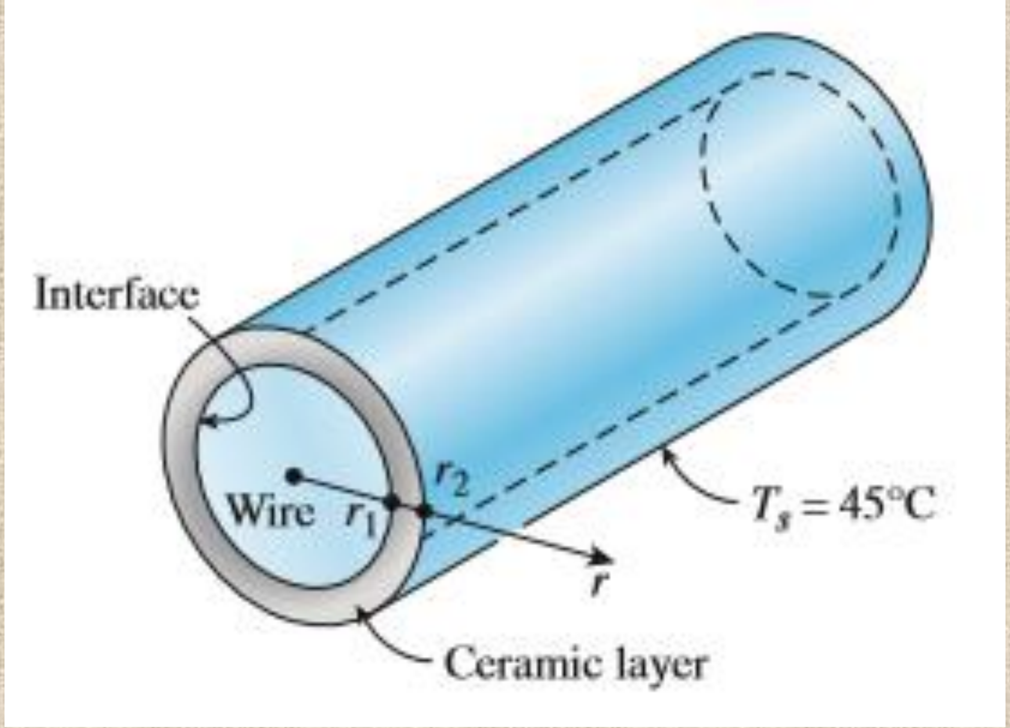
which is the desired solution for the temperature distribution in the wire as a function of r. The temperature at the centerline ($r = 0$) is obtained by replacing r in Eq. (c) by zero and substituting the known quantities,

$$T(0) = T_s + \frac{\dot{e}_{gen}}{4k} r_o^2 = 108°C + \frac{4.3 \times 10^7 W/m^3}{4 \times (13.5\ W/m{\cdot}°C)} (0.005\ m)^2 = 128°C$$

Discussion The temperature of the centerline is 20°C above the temperature of the outer surface of the wire. Note that the expression above for the centerline temperature is identical to Eq. 2–71, which was obtained using an energy balance on a control volume.

EXAMPLE 2–19 Heat Conduction in a Two-Layer Medium

Consider a long resistance wire of radius $r_1 = 0.2$ cm and thermal conductivity $k_{wire} = 15$ W/m·K in which heat is generated uniformly as a result of resistance heating at a constant rate of $\dot{e}_{gen} = 50$ W/cm^3 (Fig. 2–61). The wire is embedded in a 0.5-cm-thick layer of ceramic whose thermal conductivity is $k_{ceramic} = 1.2$ W/m·K. If the outer surface temperature of the ceramic layer is measured to be $T_s = 45°C$, determine the temperatures at the center of the resistance wire and the interface of the wire and the ceramic layer under steady conditions.



Analysis Letting T_I denote the unknown interface temperature, the heat transfer problem in the wire can be formulated as

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT_{\text{wire}}}{dr}\right)+\frac{\dot{e}_{\text{gen}}}{k}=0$$

with

$$T_{\text{wire}}(r_1)=T_I$$
$$\frac{dT_{\text{wire}}(0)}{dr}=0$$

This problem was solved in Example 2–18, and its solution was determined to be

$$T_{\text{wire}}(r)=T_I+\frac{\dot{e}_{\text{gen}}}{4k_{\text{wire}}}(r_1^2-r^2) \qquad \textbf{(a)}$$

Noting that the ceramic layer does not involve any heat generation and its outer surface temperature is specified, the heat conduction problem in that layer can be expressed as

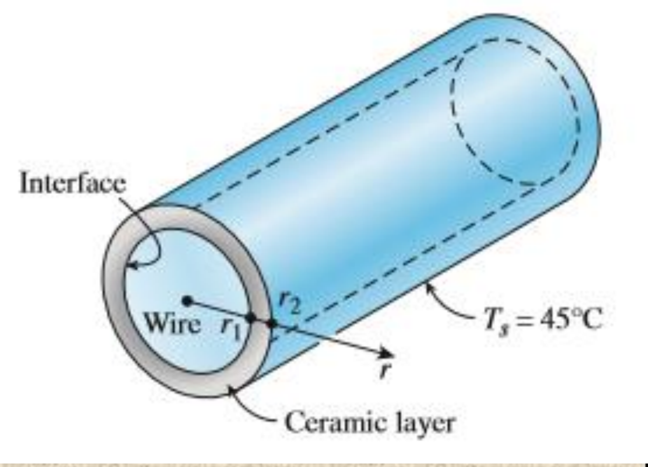
$$\frac{d}{dr}\left(r\frac{dT_{\text{ceramic}}}{dr}\right)=0$$

with

$$T_{\text{ceramic}}(r_1)=T_I$$
$$T_{\text{ceramic}}(r_2)=T_s=45^\circ\text{C}$$

This problem was solved in Example 2–16, and its solution was determined to be

$$T_{\text{ceramic}}(r)=\frac{\ln(r/r_1)}{\ln(r_2/r_1)}(T_s-T_I)+T_I \qquad \textbf{(b)}$$



We have already utilized the first interface condition by setting the wire and ceramic layer temperatures equal to T_I at the interface $r = r_1$. The interface temperature T_I is determined from the second interface condition that the heat flux in the wire and the ceramic layer at $r = r_1$ must be the same:

$$-k_{\text{wire}} \frac{dT_{\text{wire}}(r_1)}{dr} = -k_{\text{ceramic}} \frac{dT_{\text{ceramic}}(r_1)}{dr} \quad \rightarrow \quad \frac{\dot{e}_{\text{gen}} r_1}{2} = -k_{\text{ceramic}} \frac{T_s - T_I}{\ln(r_2/r_1)} \left(\frac{1}{r_1}\right)$$

Solving for T_I and substituting the given values, the interface temperature is determined to be

$$T_I = \frac{\dot{e}_{\text{gen}} r_1^2}{2k_{\text{ceramic}}} \ln \frac{r_2}{r_1} + T_s$$

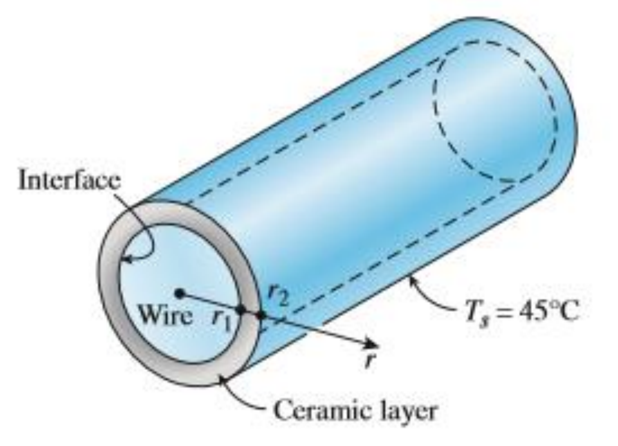
$$= \frac{(50 \times 10^6 \text{ W/m}^3)(0.002 \text{ m})^2}{2(1.2 \text{ W/m·K})} \ln \frac{0.007 \text{ m}}{0.002 \text{ m}} + 45°\text{C} = \mathbf{149.4°C}$$

Knowing the interface temperature, the temperature at the centerline ($r = 0$) is obtained by substituting the known quantities into Eq. (*a*),

$$T_{\text{wire}}(0) = T_I + \frac{\dot{e}_{\text{gen}} r_1^2}{4k_{\text{wire}}} = 149.4°\text{C} + \frac{(50 \times 10^6 \text{ W/m}^3)(0.002 \text{ m})^2}{4 \times (15 \text{ W/m·K})} = \mathbf{152.7°C}$$

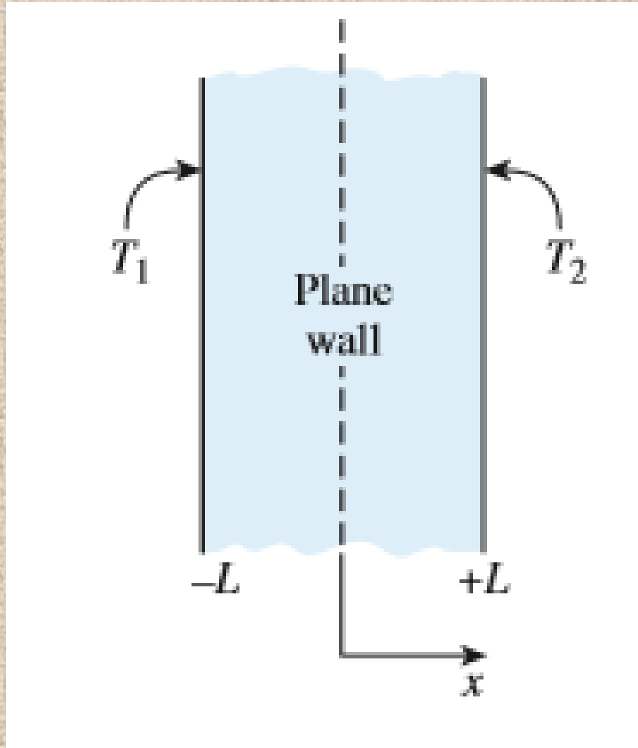
Thus the temperature of the centerline is slightly above the interface temperature.

Discussion This example demonstrates how steady one-dimensional heat conduction problems in composite media can be solved. We could also solve this problem by determining the heat flux at the interface by dividing the total heat generated in the wire by the surface area of the wire, and then using this value as the specified heat flux boundary condition for both the wire and the ceramic layer. This way the two problems are decoupled and can be solved separately.



EXAMPLE 2–20 Heat Conduction in a Plane Wall with Heat Generation

A large plane wall of thickness $2L$ experiences a uniform heat generation (Fig. 2–62). Determine the expression for the variation of temperature within the wall, if (*a*) $T_1 > T_2$ and (*b*) $T_1 = T_2$.



For steady one-dimensional heat conduction and constant thermal conductivity, the general heat conduction equation is simplified to

$$\frac{d^2T}{dx^2} + \frac{\dot{e}_{\text{gen}}}{k} = 0$$

Integrating twice gives the general solution to this second order differential equation as

$$T(x) = -\frac{\dot{e}_{\text{gen}}}{2k}x^2 + C_1x + C_2$$

(*a*) For the case of asymmetrical boundary conditions with $T_1 > T_2$, applying the boundary conditions gives

$$x = -L: \qquad T(-L) = T_1 = -\frac{\dot{e}_{\text{gen}}}{2k}L^2 - C_1L + C_2$$

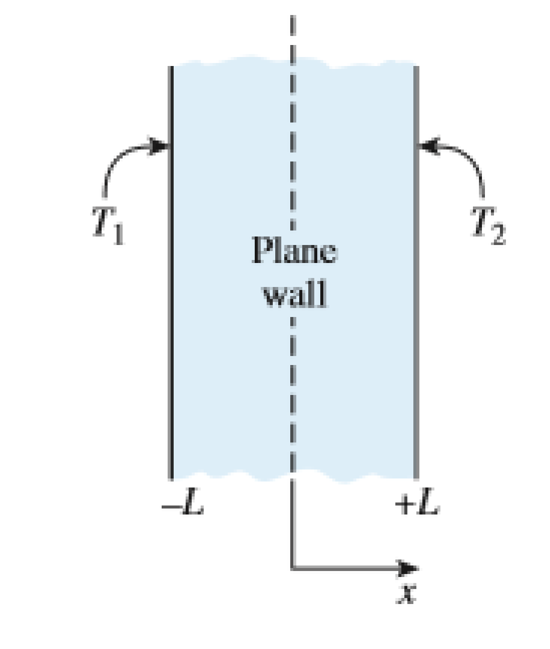
$$x = L: \qquad T(L) = T_2 = -\frac{\dot{e}_{\text{gen}}}{2k}L^2 + C_1L + C_2$$

Note that in this problem the coordinate system is placed at the middle of the plane wall ($x = 0$) and x to the right of the centerline is considered positive and to the left negative. In analyzing plane wall problems with heat generation, this notation is usually adopted in order to better capture the effect of heat generation on the temperature profile. Solving for the constants C_1 and C_2 yields

$$C_1 = \frac{T_2 - T_1}{2L} \quad \text{and} \quad C_2 = \frac{\dot{e}_{\text{gen}}}{2k}L^2 + \frac{T_1 + T_2}{2}$$

Substituting C_1 and C_2 expressions into the general solution, the variation of temperature within the wall is determined to be

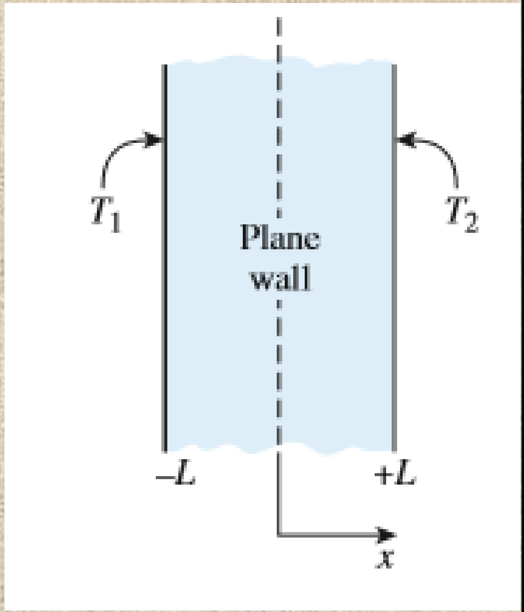
$$T(x) = \frac{\dot{e}_{\text{gen}}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{T_2 - T_1}{2}\left(\frac{x}{L}\right) + \frac{T_1 + T_2}{2} \qquad \textbf{(a)}$$



(*b*) For the case of symmetrical boundary conditions, substituting $T_2 = T_1$. into the above equation gives

$$T(x) = \frac{\dot{e}_{gen} L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_1 \qquad (b)$$

Discussion Equation (*a*) shows that the variation of temperature within the wall for the case of asymmetrical boundary conditions with $T_1 > T_2$ is not symmetric and the maximum temperature occurs to the left of the centerline. Note that Eq. (a) reduces to the temperature solution of Example 2–10 (Eq. 2–56) for heat conduction in a plane wall with no heat generation by setting $\dot{e}_{gen} = 0$ and making the appropriate coordinate transformation. In the case of symmetrical boundary conditions ($T_1 = T_2$), Eq. (*b*) shows that the variation of temperature within the wall is symmetric and the maximum temperature occurs at the centerline. This is comparable to the results shown in Example 2–17 for temperature variation in a cylindrical resistance heater.



VARIABLE THERMAL CONDUCTIVITY, *k*(*T*)

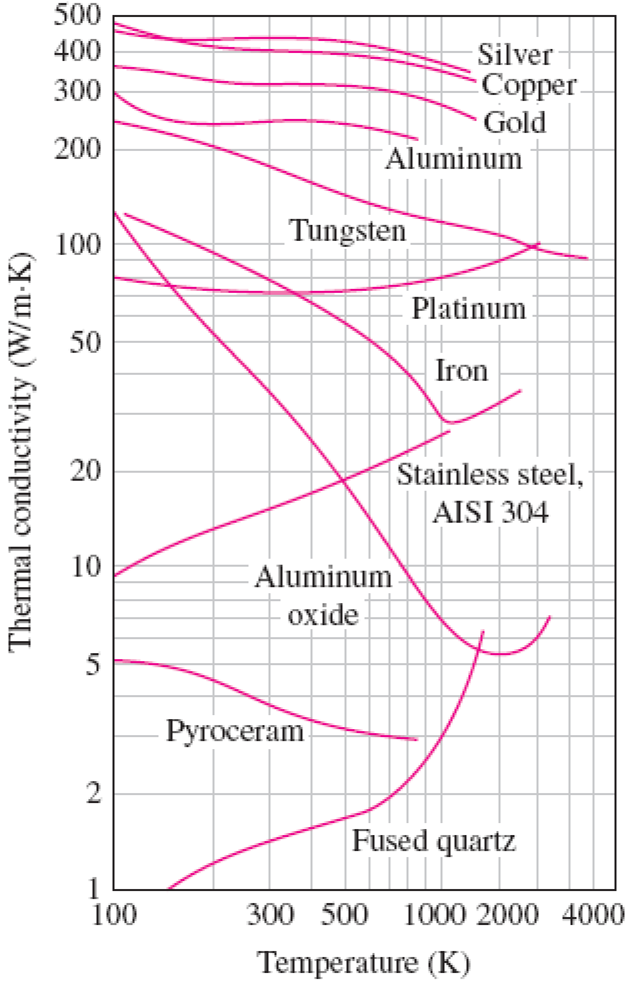


FIGURE 2–62

Variation of the thermal conductivity of some solids with temperature.

When the variation of thermal conductivity with temperature in a specified temperature interval is large, it may be necessary to account for this variation to minimize the error.

When the variation of thermal conductivity with temperature $k(T)$ is known, the average value of the thermal conductivity in the temperature range between T_1 and T_2 can be determined from

$$k_{avg} = \frac{\int_{T_1}^{T_2} k(T)dT}{T_2 - T_1}$$

$$\dot{Q}_{\text{plane wall}} = k_{avg} A \frac{T_1 - T_2}{L} = \frac{A}{L} \int_{T_2}^{T_1} k(T)dT$$

$$\dot{Q}_{\text{cylinder}} = 2\pi k_{avg} L \frac{T_1 - T_2}{\ln(r_2/r_1)} = \frac{2\pi L}{\ln(r_2/r_1)} \int_{T_2}^{T_1} k(T)dT$$

$$\dot{Q}_{\text{sphere}} = 4\pi k_{avg} r_1 r_2 \frac{T_1 - T_2}{r_2 - r_1} = \frac{4\pi r_1 r_2}{r_2 - r_1} \int_{T_2}^{T_1} k(T)dT$$

The variation in thermal conductivity of a material with temperature in the temperature range of interest can often be approximated as a linear function and expressed as

$$k(T) = k_0(1 + \beta T)$$

β temperature coefficient of thermal conductivity.

The *average* value of thermal conductivity in the temperature range T_1 to T_2 in this case can be determined from

$$k_{avg} = \frac{\int_{T_1}^{T_2} k_0(1 + \beta T)dT}{T_2 - T_1} = k_0\left(1 + \beta\frac{T_2 + T_1}{2}\right) = k(T_{avg})$$

The *average thermal conductivity* in this case is equal to the thermal conductivity value at the *average temperature.*

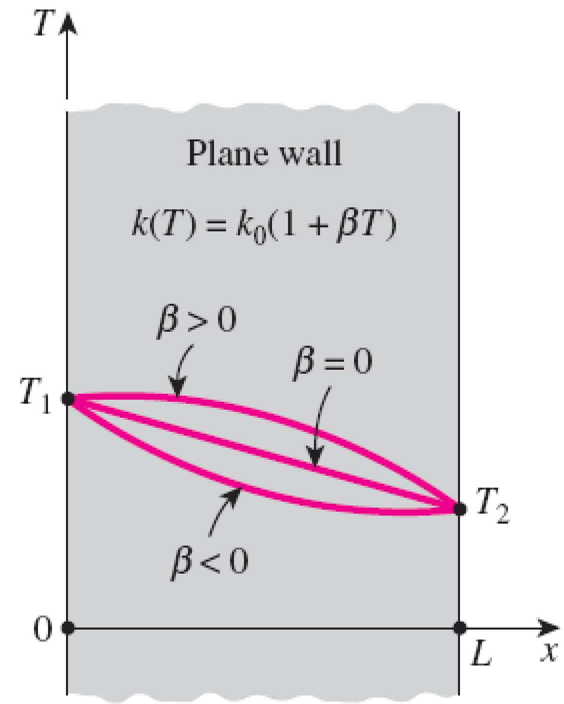


FIGURE 2–63

The variation of temperature in a plane wall during steady one-dimensional heat conduction for the cases of constant and variable thermal conductivity.

EXAMPLE 2–21 Variation of Temperature in a Wall with $k(T)$

Consider a plane wall of thickness L whose thermal conductivity varies linearly in a specified temperature range as $k(T) = k_0(1 + \beta T)$ where k_0 and β are constants. The wall surface at $x = 0$ is maintained at a constant temperature of T_1 while the surface at $x = L$ is maintained at T_2, as shown in Fig. 2–65. Assuming steady one-dimensional heat transfer, obtain a relation for (*a*) the heat transfer rate through the wall and (*b*) the temperature distribution $T(x)$ in the wall.

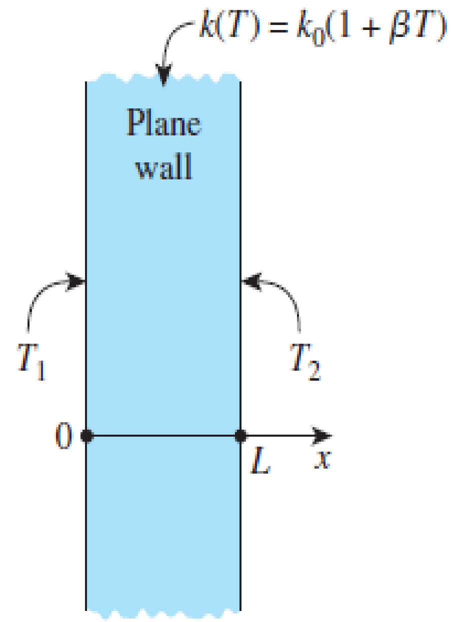


FIGURE 2–65
Schematic for Example 2–21.

SOLUTION A plate with variable conductivity is subjected to specified temperatures on both sides. The variation of temperature and the rate of heat transfer are to be determined.

Assumptions **1** Heat transfer is given to be steady and one-dimensional. **2** Thermal conductivity varies linearly. **3** There is no heat generation.

Properties The thermal conductivity is given to be $k(T) = k_0(1 + \beta T)$.

Analysis (*a*) The rate of heat transfer through the wall can be determined from

$$\dot{Q} = k_{avg} A \frac{T_1 - T_2}{L}$$

where A is the heat conduction area of the wall and

$$k_{avg} = k(T_{avg}) = k_0\left(1 + \beta \frac{T_2 + T_1}{2}\right)$$

is the average thermal conductivity (Eq. 2–80).

(*b*) To determine the temperature distribution in the wall, we begin with Fourier's law of heat conduction, expressed as

$$\dot{Q} = -k(T)\, A \frac{dT}{dx}$$

where the rate of conduction heat transfer $\dot{Q}$ and the area A are constant. Separating variables and integrating from $x = 0$ where $T(0) = T_1$ to any x where $T(x) = T$, we get

$$\int_0^x \dot{Q}\, dx = -A \int_{T_1}^{T} k(T) dT$$

Substituting $k(T) = k_0(1 + \beta T)$ and performing the integrations we obtain

$$\dot{Q}x = -Ak_0[(T - T_1) + \beta(T^2 - T_1^2)/2]$$

Substituting the $\dot{Q}$ expression from part (*a*) and rearranging give

$$T^2 + \frac{2}{\beta}T + \frac{2k_{\text{avg}}}{\beta k_0}\frac{x}{L}(T_1 - T_2) - T_1^2 - \frac{2}{\beta}T_1 = 0$$

which is a *quadratic* equation in the unknown temperature T. Using the quadratic formula, the temperature distribution $T(x)$ in the wall is determined to be

$$T(x) = -\frac{1}{\beta} \pm \sqrt{\frac{1}{\beta^2} - \frac{2k_{\text{avg}}}{\beta k_0}\frac{x}{L}(T_1 - T_2) + T_1^2 + \frac{2}{\beta}T_1}$$

Discussion The proper sign of the square root term (+ or −) is determined from the requirement that the temperature at any point within the medium must remain between T_1 and T_2. This result explains why the temperature distribution in a plane wall is no longer a straight line when the thermal conductivity varies with temperature.

EXAMPLE 2–22 Heat Conduction through a Wall with $k(T)$

Consider a 2-m-high and 0.7-m-wide bronze plate whose thickness is 0.1 m. One side of the plate is maintained at a constant temperature of 600 K while the other side is maintained at 400 K, as shown in Fig. 2–66. The thermal conductivity of the bronze plate can be assumed to vary linearly in that temperature range as $k(T) = k_0(1 + \beta T)$ where $k_0 = 38$ W/m·K and $\beta = 9.21 \times 10^{-4}$ K^{-1}. Disregarding the edge effects and assuming steady one-dimensional heat transfer, determine the rate of heat conduction through the plate.

SOLUTION A plate with variable conductivity is subjected to specified temperatures on both sides. The rate of heat transfer is to be determined.

Assumptions **1** Heat transfer is given to be steady and one-dimensional. **2** Thermal conductivity varies linearly. **3** There is no heat generation.

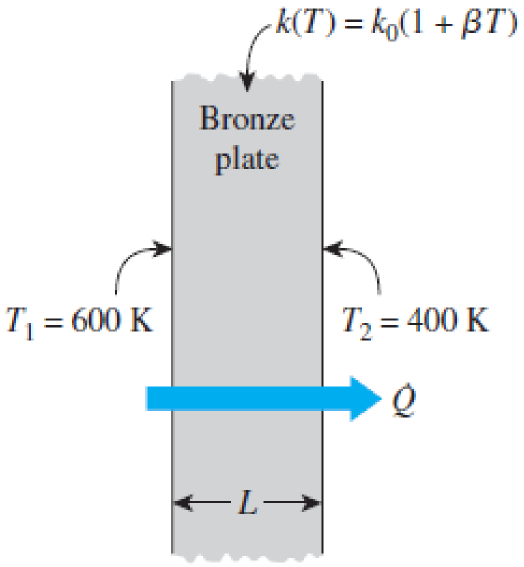


FIGURE 2–66
Schematic for Example 2–22.

Properties The thermal conductivity is given to be $k(T) = k_0(1 + \beta T)$.

Analysis The average thermal conductivity of the medium in this case is simply the value at the average temperature and is determined from

$$k_{avg} = k(T_{avg}) = k_0\left(1 + \beta\frac{T_2 + T_1}{2}\right)$$

$$= (38 \text{ W/m·K})\left[1 + (9.21 \times 10^{-4} \text{ K}^{-1})\frac{(600 + 400) \text{ K}}{2}\right]$$

$$= 55.5 \text{ W/m·K}$$

$$\dot{Q} = k_{avg} A \frac{T_1 - T_2}{L}$$

$$= (55.5 \text{ W/m·K})(2 \text{ m} \times 0.7 \text{ m})\frac{(600 - 400)\text{K}}{0.1 \text{ m}} = 155 \text{ kW}$$

Discussion We would have obtained the same result by substituting the given $k(T)$ relation into the second part of Eq. 2–76 and performing the indicated integration.

Summary

- Introduction
  - ✓ Steady versus Transient Heat Transfer
  - ✓ Multidimensional Heat Transfer
  - ✓ Heat Generation
- One-Dimensional Heat Conduction Equation
  - ✓ Heat Conduction Equation in a Large Plane Wall
  - ✓ Heat Conduction Equation in a Long Cylinder
  - ✓ Heat Conduction Equation in a Sphere
  - ✓ Combined One-Dimensional Heat Conduction Equation
- General Heat Conduction Equation
  - ✓ Rectangular Coordinates
  - ✓ Cylindrical Coordinates
  - ✓ Spherical Coordinates
- Boundary and Initial Conditions
- Solution of Steady One-Dimensional Heat Conduction Problems
- Heat Generation in a Solid
- Variable Thermal Conductivity $k(T)$