Chapter 2: Heat Conduction Equation

Objectives

When you finish studying this chapter, you should be able to:

- Understand multidimensionality and time dependence of heat transfer, and the conditions under which a heat transfer problem can be approximated as being one-dimensional,
- Obtain the differential equation of heat conduction in various coordinate systems, and simplify it for steady one-dimensional case,
- Identify the thermal conditions on surfaces, and express them mathematically as boundary and initial conditions,
- Solve one-dimensional heat conduction problems and obtain the temperature distributions within a medium and the heat flux,
- Analyze one-dimensional heat conduction in solids that involve heat generation, and
- Evaluate heat conduction in solids with temperature-dependent thermal conductivity.

Introduction

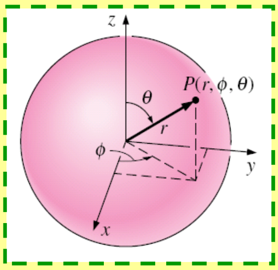
- Although heat transfer and temperature are closely related, they are of a different nature.
- Temperature has only magnitude

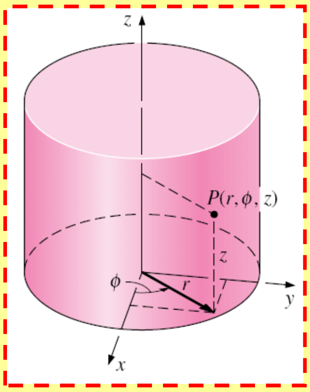
  it is a *scalar* quantity.
- Heat transfer has direction as well as magnitude

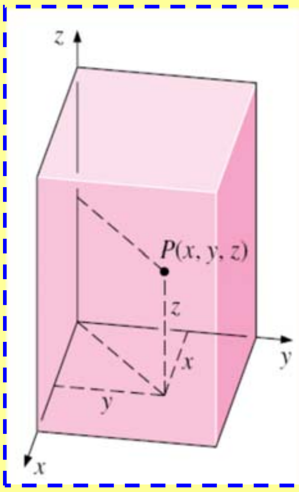
  it is a *vector* quantity.
- We work with a coordinate system and indicate direction with plus or minus signs.

Introduction ─ Continue

- The driving force for any form of heat transfer is the *temperature difference.*
- The larger the temperature difference, the larger the rate of heat transfer.
- Three prime coordinate systems:
  - rectangular ($T(x, y, z, t)$) ,
  - cylindrical ($T(r, \phi, z, t)$),
  - spherical ($T(r, \phi, \theta, t)$).







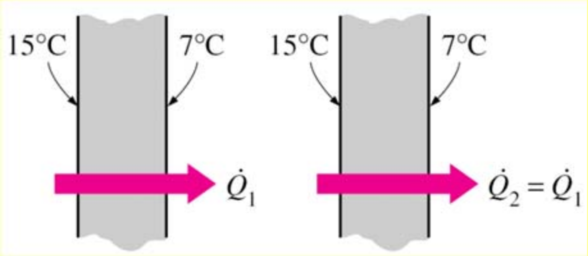
Introduction ─ Continue

Classification of conduction heat transfer problems:

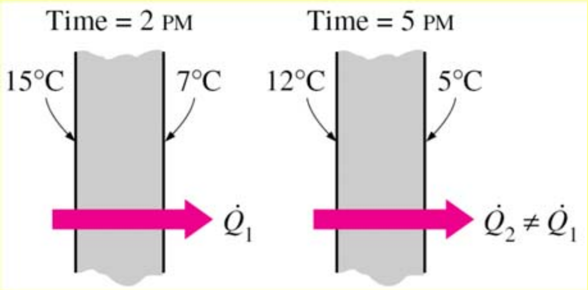
- steady versus transient heat transfer,
- multidimensional heat transfer,
- heat generation.

Steady versus Transient Heat Transfer

- ***Steady*** implies *no change* with time at any point within the medium



- ***Transient*** implies *variation with time* or *time dependence*

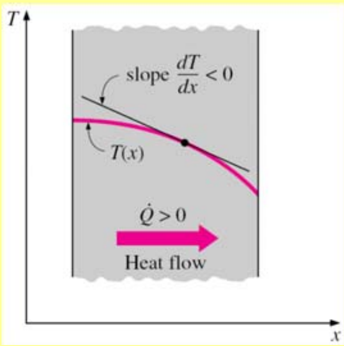


Multidimensional Heat Transfer

- Heat transfer problems are also classified as being:
  - *one-dimensional*,
  - *two dimensional*,
  - *three-dimensional*.
- In the most general case, heat transfer through a medium is **three-dimensional**. However, some problems can be classified as two- or one-dimensional depending on the relative magnitudes of heat transfer rates in different directions and the level of accuracy desired.

- The rate of heat conduction through a medium in a specified direction (say, in the *x*-direction) is expressed by **Fourier's law of heat conduction** for one-dimensional heat conduction as:

$$\dot{Q}_{cond} = -kA\frac{dT}{dx} \qquad \text{(W)} \qquad \textbf{(2-1)}$$

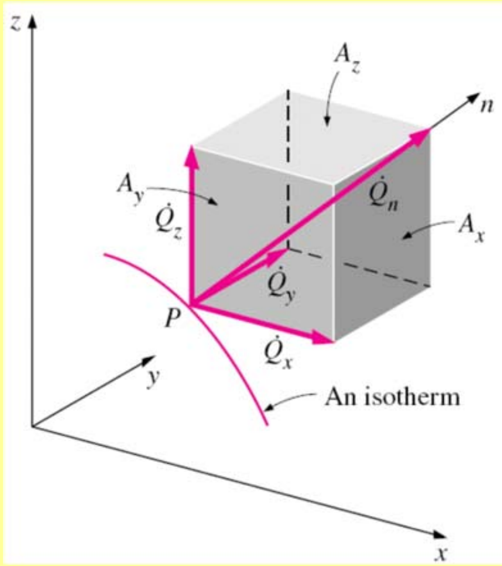


- Heat is conducted in the direction of decreasing temperature, and thus the temperature gradient is negative when heat is conducted in the positive *x*-direction.

General Relation for Fourier's Law of Heat Conduction

- The heat flux vector at a point *P* on the surface of the figure must be perpendicular to the surface, and it must point in the direction of decreasing temperature
- If *n* is the normal of the isothermal surface at point *P*, the rate of heat conduction at that point can be expressed by Fourier's law as

$$\dot{Q}_n = -kA\frac{dT}{dn} \qquad \text{(W)} \qquad \textbf{(2-2)}$$



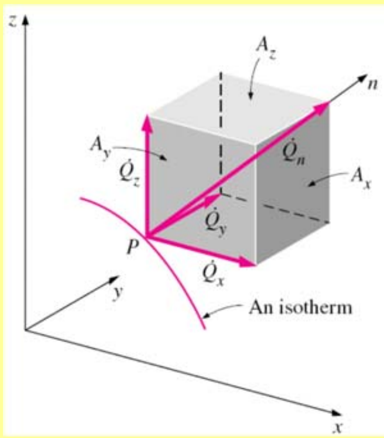
General Relation for Fourier's Law of Heat Conduction-Continue

- In rectangular coordinates, the heat conduction vector can be expressed in terms of its components as

$$\vec{\dot{Q}}_n = \dot{Q}_x\vec{i} + \dot{Q}_y\vec{j} + \dot{Q}_z\vec{k} \qquad \textbf{(2-3)}$$

- which can be determined from Fourier's law as

$$\begin{cases} \dot{Q}_x = -kA_x\dfrac{\partial T}{\partial x} \\ \dot{Q}_y = -kA_y\dfrac{\partial T}{\partial y} \\ \dot{Q}_z = -kA_z\dfrac{\partial T}{\partial z} \end{cases} \qquad \textbf{(2-4)}$$



Heat Generation

- Examples:
  - electrical energy being converted to heat at a rate of I^2R,
  - fuel elements of nuclear reactors,
  - exothermic chemical reactions.
- Heat generation is a *volumetric phenomenon*.
- The rate of heat generation units : W/m^3 or $Btu/h \cdot ft^3$.
- The rate of heat generation in a medium may vary with time as well as position within the medium.
- The *total* rate of heat generation in a medium of volume V can be determined from

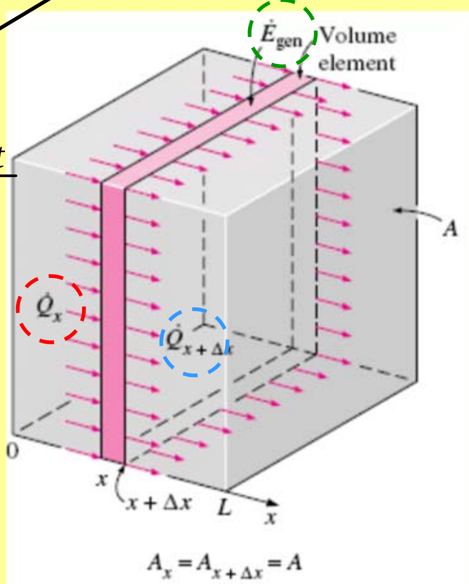
$$\dot{E}_{gen} = \int_V \dot{e}_{gen} dV \qquad (W) \qquad \textbf{(2-5)}$$

One-Dimensional Heat Conduction Equation - Plane Wall

$$\left(\begin{array}{c}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x\end{array}\right) - \left(\begin{array}{c}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x+\Delta x\end{array}\right) + \left(\begin{array}{c}\text{Rate of heat}\\ \text{generation inside}\\ \text{the element}\end{array}\right) = \left(\begin{array}{c}\text{Rate of change of}\\ \text{the energy content}\\ \text{of the element}\end{array}\right)$$

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{E}_{gen,element} = \frac{\Delta E_{element}}{\Delta t}$$

(2-6)



$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{E}_{gen,element} = \frac{\Delta E_{element}}{\Delta t} \qquad \textbf{(2-6)}$$

- The change in the energy content and the rate of heat generation can be expressed as

$$\begin{cases} \Delta E_{element} = E_{t+\Delta t} - E_t = mc\left(T_{t+\Delta t} - T_t\right) = \rho c A \Delta x \left(T_{t+\Delta t} - T_t\right) & \textbf{(2-7)} \\ \dot{E}_{gen,element} = \dot{e}_{gen} V_{element} = \dot{e}_{gen} A \Delta x & \textbf{(2-8)} \end{cases}$$

- Substituting into Eq. 2–6, we get

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{e}_{gen} A \Delta x = \rho c A \Delta x \frac{T_{t+\Delta t} - T_t}{\Delta t} \qquad \textbf{(2-9)}$$

- Dividing by $A\Delta x$, taking the limit as $\Delta x \rightarrow 0$ and $\Delta t \rightarrow 0$, and from Fourier's law:

$$\frac{1}{A}\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) + \dot{e}_{gen} = \rho c \frac{\partial T}{\partial t} \qquad \textbf{(2-11)}$$

The area A is constant for a plane wall → the one dimensional transient heat conduction equation in a plane wall is

Variable conductivity: $$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{e}_{gen} = \rho c \frac{\partial T}{\partial t} \qquad \textbf{(2-13)}$$

Constant conductivity: $$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{e}_{gen}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad ; \quad \alpha = \frac{k}{\rho c} \qquad \textbf{(2-14)}$$

The one-dimensional conduction equation may be reduces to the following forms under special conditions

1) Steady-state: $$\frac{d^2 T}{dx^2} + \frac{\dot{e}_{gen}}{k} = 0 \qquad \textbf{(2-15)}$$

2) Transient, no heat generation: $$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \textbf{(2-16)}$$

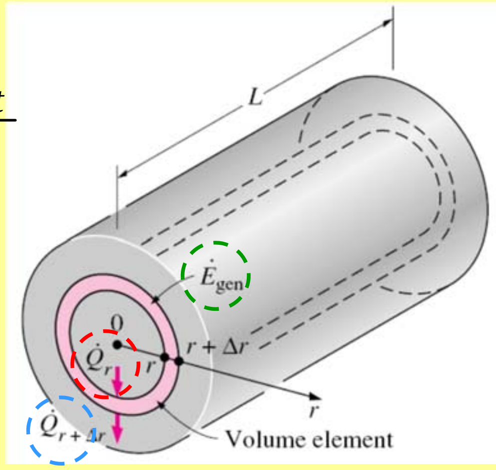
3) Steady-state, no heat generation: $$\frac{d^2 T}{dx^2} = 0 \qquad \textbf{(2-17)}$$

One-Dimensional Heat Conduction Equation - Long Cylinder

$$\begin{pmatrix}\text{Rate of heat}\\\text{conduction}\\\text{at } r\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\\text{conduction}\\\text{at } r+\Delta r\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\\text{generation inside}\\\text{the element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change of}\\\text{the energy content}\\\text{of the element}\end{pmatrix}$$

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{E}_{gen,element} = \frac{\Delta E_{element}}{\Delta t}$$

(2-18)



$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{E}_{gen,element} = \frac{\Delta E_{element}}{\Delta t} \qquad \textbf{(2-18)}$$

- The change in the energy content and the rate of heat generation can be expressed as

$$\begin{cases}\Delta E_{element} = E_{t+\Delta t} - E_t = mc\left(T_{t+\Delta t} - T_t\right) = \rho c A \Delta r\left(T_{t+\Delta t} - T_t\right) & \textbf{(2-19)}\\ \dot{E}_{gen,element} = \dot{e}_{gen} V_{element} = \dot{e}_{gen} A \Delta r & \textbf{(2-20)}\end{cases}$$

- Substituting into Eq. 2–18, we get

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{e}_{gen} A \Delta r = \rho c A \Delta r \frac{T_{t+\Delta t} - T_t}{\Delta t} \qquad \textbf{(2-21)}$$

- Dividing by $A\Delta r$, taking the limit as $\Delta r \rightarrow 0$ and $\Delta t \rightarrow 0$, and from Fourier's law:

$$\frac{1}{A}\frac{\partial}{\partial r}\left(kA\frac{\partial T}{\partial r}\right) + \dot{e}_{gen} = \rho c \frac{\partial T}{\partial t} \qquad \textbf{(2-23)}$$

Noting that the area varies with the independent variable r according to $A=2\pi rL$, the one dimensional transient heat conduction equation in a plane wall becomes

Variable conductivity: $$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)+\dot{e}_{gen}=\rho c\frac{\partial T}{\partial t} \quad \textbf{(2-25)}$$

Constant conductivity: $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\dot{e}_{gen}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \textbf{(2-26)}$$

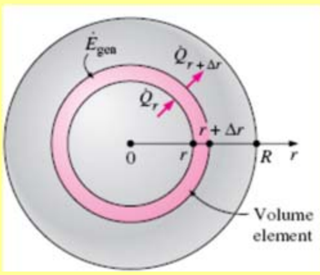
The one-dimensional conduction equation may be reduces to the following forms under special conditions

1) Steady-state: $$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{e}_{gen}}{k}=0 \quad \textbf{(2-27)}$$

2) Transient, no heat generation: $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \textbf{(2-28)}$$

3) Steady-state, no heat generation: $$\frac{d}{dr}\left(r\frac{dT}{dr}\right)=0 \quad \textbf{(2-29)}$$

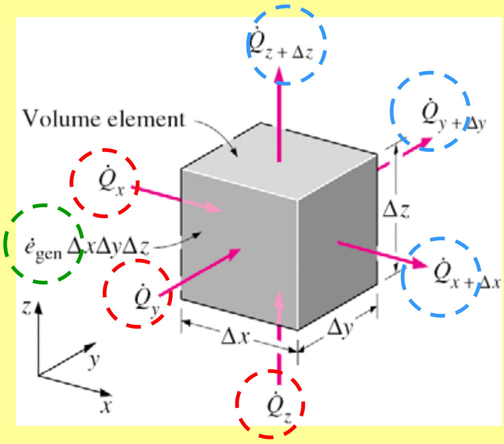
One-Dimensional Heat Conduction Equation - Sphere



Variable conductivity: $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2k\frac{\partial T}{\partial r}\right)+\dot{e}_{gen}=\rho c\frac{\partial T}{\partial t} \quad \textbf{(2-30)}$$

Constant conductivity: $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{\dot{e}_{gen}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \textbf{(2-31)}$$

General Heat Conduction Equation



Rate of heat conduction at *x, y,* and *z* - **Rate of heat conduction at *x+Δx, y+Δy, and z+Δz*** + **Rate of heat generation inside the element** = **Rate of change of the energy content of the element**

$$\dot{Q}_x + \dot{Q}_y + \dot{Q}_z - \dot{Q}_{x+\Delta x} - \dot{Q}_{y+\Delta y} - \dot{Q}_{z+\Delta z} + E_{gen,element} = \frac{\Delta E_{element}}{\Delta t} \quad \textbf{(2-36)}$$

Repeating the mathematical approach used for the one-dimensional heat conduction the three-dimensional heat conduction equation is determined to be

Constant conductivity:

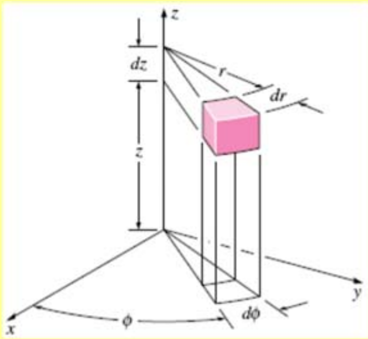
$$\underbrace{\overbrace{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}}^{\text{Two-dimensional}} + \frac{\partial^2 T}{\partial z^2}}_{\text{Three-dimensional}} + \frac{\dot{e}_{gen}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \textbf{(2-39)}$$

1) Steady-state:
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{e}_{gen}}{k} = 0 \quad \textbf{(2-40)}$$

2) Transient, no heat generation:
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \textbf{(2-41)}$$

3) Steady-state, no heat generation:
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \quad \textbf{(2-42)}$$

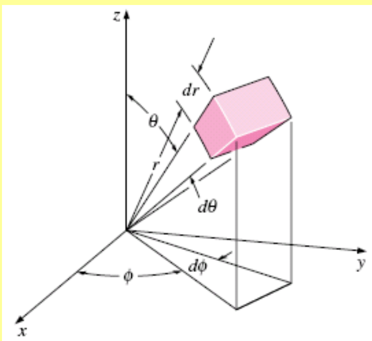
Cylindrical Coordinates



$$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial T}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{e}_{gen}=\rho c\frac{\partial T}{\partial t}$$

(2-43)

Spherical Coordinates



$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\dot{e}_{gen}=\rho c\frac{\partial T}{\partial t}$$

(2-44)

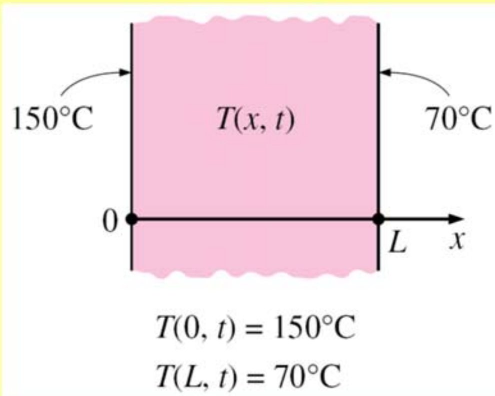
Boundary and Initial Conditions

- Specified Temperature Boundary Condition
- Specified Heat Flux Boundary Condition
- Convection Boundary Condition
- Radiation Boundary Condition
- Interface Boundary Conditions
- Generalized Boundary Conditions

Specified Temperature Boundary Condition

For one-dimensional heat transfer through a plane wall of thickness L, for example, the specified temperature boundary conditions can be expressed as

$$T(0, t) = T_1$$
$$T(\mathrm{L}, t) = T_2 \qquad \textbf{(2-46)}$$

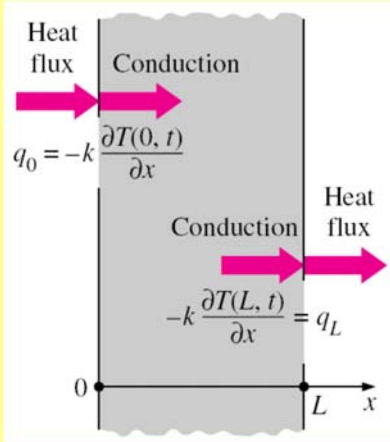


The specified temperatures can be constant, which is the case for steady heat conduction, or may vary with time.

Specified Heat Flux Boundary Condition

The heat flux in the positive *x*-direction anywhere in the medium, including the boundaries, can be expressed by *Fourier's law* of heat conduction as

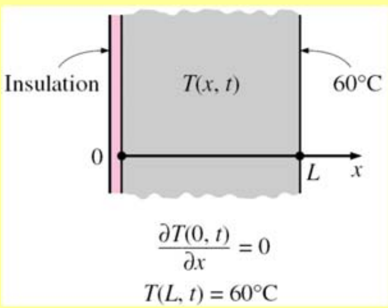
$$\dot{q} = -k\frac{dT}{dx} = \left(\textbf{Heat flux in the positive x-direction} \right) \quad \textbf{(2-47)}$$



The sign of the specified heat flux is determined by inspection: *positive* if the heat flux is in the positive direction of the coordinate axis, and *negative* if it is in the opposite direction.

Two Special Cases

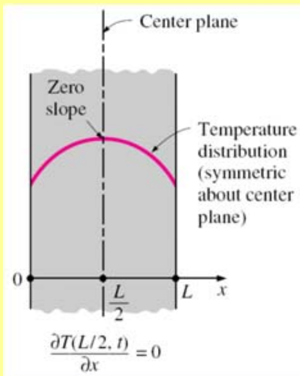
Insulated boundary



$$k\frac{\partial T(0,t)}{\partial x} = 0 \quad \text{or} \quad \frac{\partial T(0,t)}{\partial x} = 0$$

(2-49)

Thermal symmetry

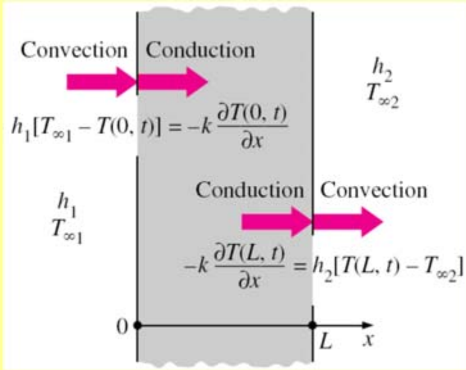


$$\frac{\partial T\left(L/2, t\right)}{\partial x} = 0$$

(2-50)

Convection Boundary Condition

Heat conduction at the surface in a selected direction = **Heat convection at the surface in the same direction**



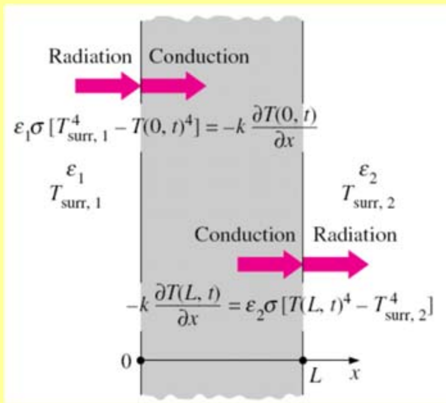
$$-k \frac{\partial T(0,t)}{\partial x} = h_1 \left[T_{\infty 1} - T(0,t) \right] \qquad \textbf{(2-51a)}$$

and

$$-k \frac{\partial T(L,t)}{\partial x} = h_2 \left[T(L,t) - T_{\infty 2} \right] \qquad \textbf{(2-51b)}$$

Radiation Boundary Condition

Heat conduction at the surface in a selected direction = **Radiation exchange at the surface in the same direction**



$$-k \frac{\partial T(0,t)}{\partial x} = \varepsilon_1 \sigma \left[T^4_{surr,1} - T(0,t)^4 \right] \qquad \textbf{(2-52a)}$$

and

$$-k \frac{\partial T(L,t)}{\partial x} = \varepsilon_2 \sigma \left[T(L,t)^4 - T^4_{surr,2} \right] \qquad \textbf{(2-52b)}$$

Interface Boundary Conditions

At the interface the requirements are:

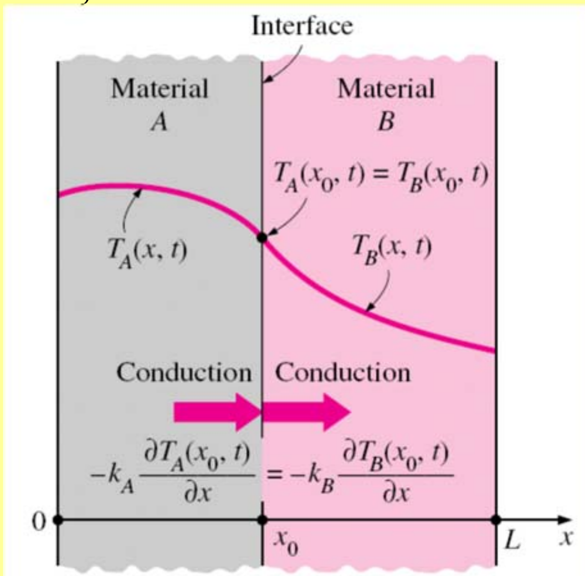
(1) two bodies in contact must have the *same temperature* at the area of contact,

(2) an interface (which is a surface) cannot store any energy, and thus the *heat flux* on the two sides of an interface *must be the same*.

$$T_A(x_0, t) = T_B(x_0, t) \qquad \textbf{(2-53)}$$

and

$$-k_A \frac{\partial T_A(x_0,t)}{\partial x} = -k_B \frac{\partial T_B(x_0,t)}{\partial x} \qquad \textbf{(2-54)}$$



Generalized Boundary Conditions

In general a surface may involve convection, radiation, *and* specified heat flux simultaneously. The boundary condition in such cases is again obtained from a surface energy balance, expressed as

$$\left(\begin{array}{c}\textbf{Heat transfer}\\ \textbf{to the surface}\\ \textbf{in all modes}\end{array}\right) = \left(\begin{array}{c}\textbf{Heat transfer}\\ \textbf{from the surface}\\ \textbf{In all modes}\end{array}\right)$$

Heat Generation in Solids

The quantities of major interest in a medium with heat generation are the surface temperature T_s and the maximum temperature T_{max} that occurs in the medium in *steady* operation.

Heat Generation in Solids -The Surface Temperature

$$\left(\text{Rate of heat transfer from the solid}\right) = \left(\text{Rate of energy generation within the solid}\right) \qquad \textbf{(2-63)}$$

For *uniform* heat generation within the medium

$$\dot{Q} = \dot{e}_{gen} V \qquad \text{(W)} \qquad \textbf{(2-64)}$$

The heat transfer rate by convection can also be expressed from Newton's law of cooling as

$$\dot{Q} = hA_s \left(T_s - T_\infty\right) \qquad \text{(W)} \qquad \textbf{(2-65)}$$

-

$$T_s = T_\infty + \frac{\dot{e}_{gen} V}{hA_s} \qquad \textbf{(2-66)}$$

Heat Generation in Solids -The Surface Temperature

For a large *plane wall* of thickness $2L$ ($A_s=2A_{wall}$ and $V=2LA_{\mathrm{wall}}$)

$$T_{s,\,plane\ wall} = T_\infty + \frac{\dot{e}_{gen} L}{h} \qquad \textbf{(2-67)}$$

For a long solid *cylinder* of radius r_0 ($A_s=2\pi r_0 L$ and $V=\pi r_0^2 L$)

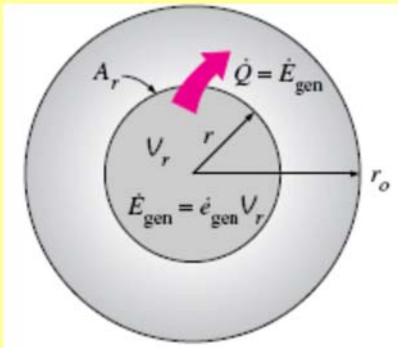
$$T_{s,cylinder} = T_\infty + \frac{\dot{e}_{gen} r_0}{2h} \qquad \textbf{(2-68)}$$

For a solid *sphere* of radius r_0 ($A_s=4\pi r_0^2$ and $V={}^4/_3\pi r_0^3$)

$$T_{s,sphere} = T_\infty + \frac{\dot{e}_{gen} r_0}{3h} \qquad \textbf{(2-69)}$$

Heat Generation in Solids -The maximum Temperature in a Cylinder (the Centerline)

The *heat generated* within an inner cylinder must be equal to the *heat conducted* through its outer surface.



$$-kA_r \frac{dT}{dr} = \dot{e}_{gen} V_r \qquad \textbf{(2-70)}$$

Substituting these expressions into the above equation and separating the variables, we get

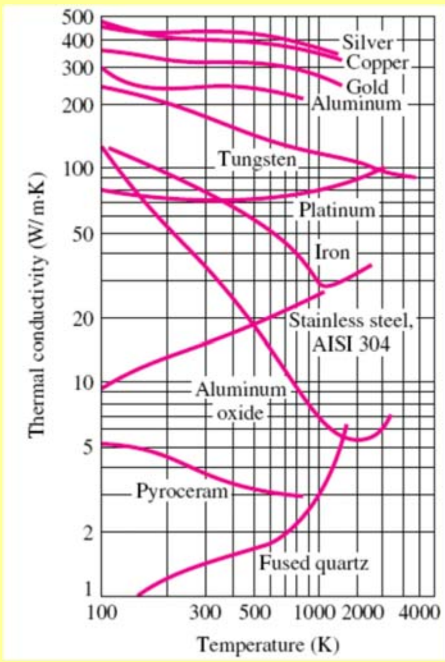
$$-k\left(2\pi rL\right)\frac{dT}{dr} = \dot{e}_{gen}\left(\pi r^2 L\right) \rightarrow dT = -\frac{\dot{e}_{gen}}{2k} r dr$$

Integrating from r =0 where $T(0) = T_0$ to $r = r_o$

$$\Delta T_{\max,cylinder} = T_0 - T_s = \frac{\dot{e}_{gen} r_0^2}{4k} \qquad \textbf{(2-71)}$$

Variable Thermal Conductivity, $k(T)$

- The thermal conductivity of a material, in general, varies with temperature.
- An average value for the thermal conductivity is commonly used when the variation is mild.
- This is also common practice for other temperature-dependent properties such as the density and specific heat.



Variable Thermal Conductivity for One-Dimensional Cases

When the variation of thermal conductivity with temperature $k(T)$ is known, the average value of the thermal conductivity in the temperature range between T_1 and T_2 can be determined from

$$k_{ave} = \frac{\int_{T_1}^{T_2} k(T)dT}{T_2 - T_1} \qquad \textbf{(2-75)}$$

The variation in thermal conductivity of a material with can often be approximated as a linear function and expressed as

$$k(T) = k_0(1 + \beta T) \qquad \textbf{(2-79)}$$

β the **temperature coefficient of thermal conductivity**.

Variable Thermal Conductivity

- For a plane wall the temperature varies linearly during steady one-dimensional heat conduction when the thermal conductivity is constant.
- This is no longer the case when the thermal conductivity changes with temperature (even linearly).