Chapter 3: Pressure and Fluid Statics

Pressure

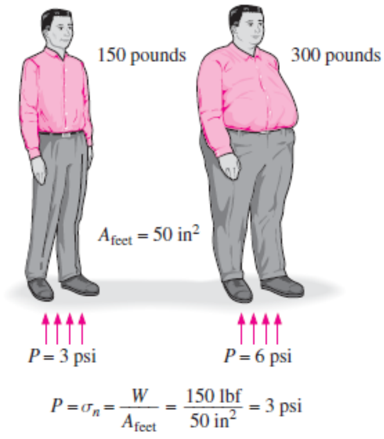
- **Pressure** is defined as a *normal force exerted by a fluid per unit area*.
- ***We*** speak of pressure only when we deal with a gas or a liquid. The counterpart of pressure in solids is *normal stress.*
- Units of pressure are N/m^2, which is called a **pascal** (Pa).
- Since the unit Pa is too small for pressures encountered in practice, *kilopascal* (1 kPa = 10^3 Pa) and *megapascal* (1 MPa = 10^6 Pa) are commonly used.
- Other units include *bar*, *atm*, kgf/cm^2, $lbf/in^2 = psi$.

$$1 \text{ bar} = 10^5 \text{ Pa} = 0.1 \text{ MPa} = 100 \text{ kPa}$$

$$1 \text{ atm} = 101{,}325 \text{ Pa} = 101.325 \text{ kPa} = 1.01325 \text{ bars}$$

Pressure

- Pressure is also used for solids as synonymous to *normal stress, which is* force acting perpendicular to the surface per unit area.
- This also explains how a person can walk on fresh snow without sinking by wearing large snowshoes, and how a person cuts with little effort when using a sharp knife.



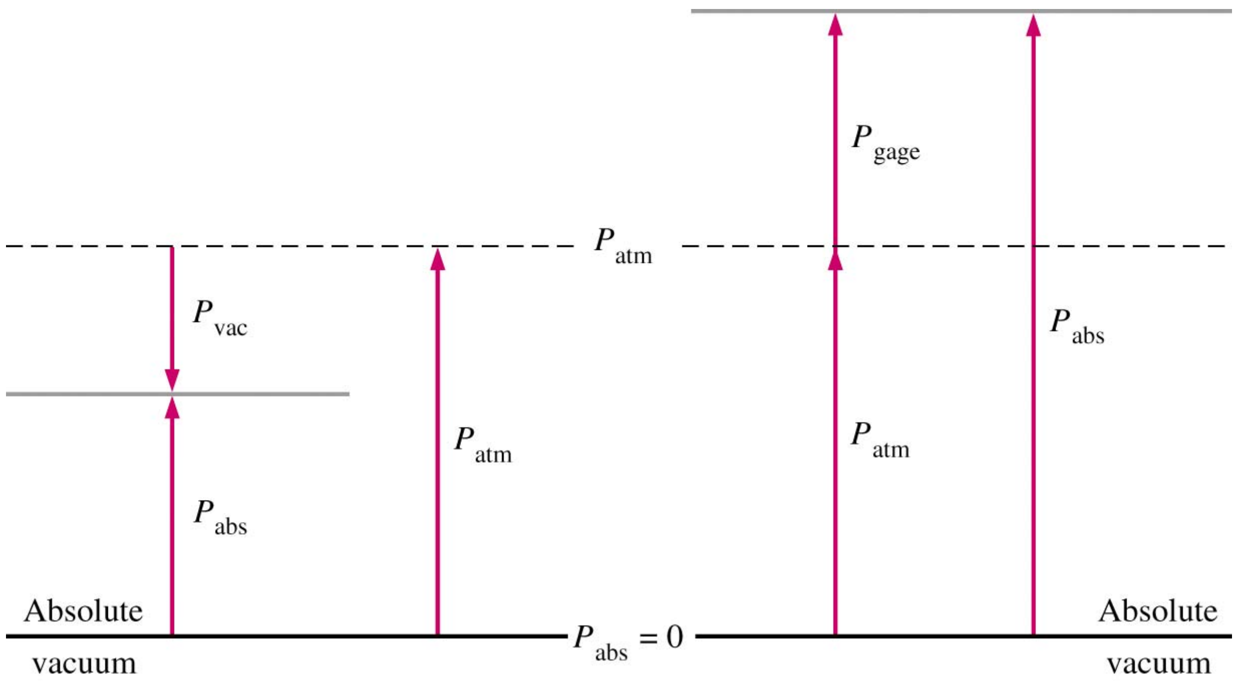
Absolute, gage, and vacuum pressures

- Actual pressure at a give point is called the **absolute pressure**.
- Most pressure-measuring devices are calibrated to read zero in the atmosphere, and therefore indicate **gage pressure**, $P_{gage} = P_{abs} - P_{atm}$.
- Pressure below atmospheric pressure are called **vacuum pressure**,

$$P_{vac} = P_{atm} - P_{abs}$$

Some basic pressure gages.

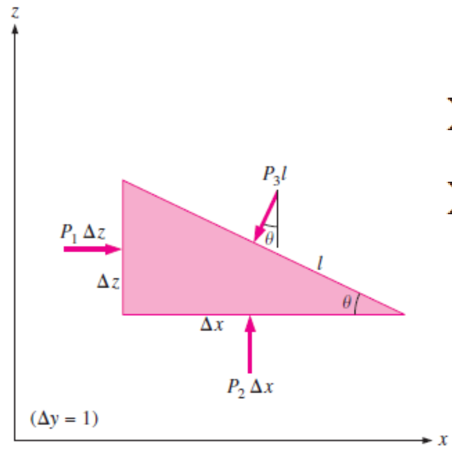
Absolute, gage, and vacuum pressures



Pressure at a Point

- Pressure is the *compressive force per unit area*
- Pressure at any point in a fluid is the same in all directions.
- Pressure has a magnitude, but not a specific direction, and thus it is a scalar quantity.

Pressure at a Point



Forces acting on a wedge-shaped fluid element in equilibrium.

From Newton's second law, a force balance in the *x- and z directions* gives

$$\sum F_x = ma_x = 0: \qquad P_1\,\Delta z - P_3 l \sin\theta = 0$$

$$\sum F_z = ma_z = 0: \qquad P_2\,\Delta x - P_3 l \cos\theta - \frac{1}{2}\rho g\,\Delta x\,\Delta z = 0$$

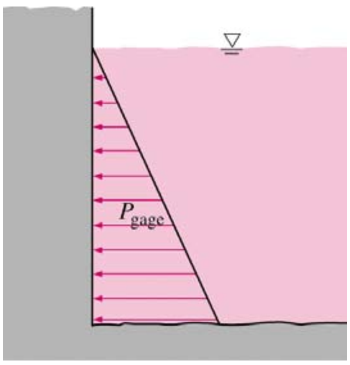
$$\Rightarrow \quad P_1 - P_3 = 0$$

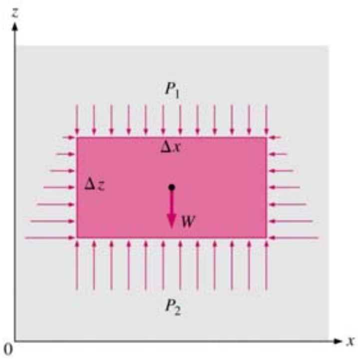
$$P_2 - P_3 - \frac{1}{2}\rho g\,\Delta z = 0$$

$$\Delta z \to 0 \quad \Rightarrow \quad P_1 = P_2 = P_3 = P$$

Thus we conclude that the pressure at a point in a fluid has the same magnitude in all directions. It can be shown in the absence of shear forces that this result is applicable to fluids in motion as well as fluids at rest.

Variation of Pressure with Depth





- In the presence of a gravitational field, pressure increases with depth because more fluid rests on deeper layers.
- To obtain a relation for the variation of pressure with depth, consider rectangular element $\quad \sum F_z = ma_z = 0$

$$P_2\Delta x - P_1\Delta x - \rho g\Delta x\Delta z = 0$$

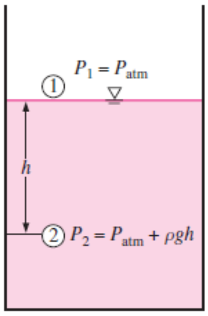
  - Force balance in z-direction gives
  - Dividing by Δx and rearranging gives

$$\boxed{\Delta P = P_2 - P_1 = \rho g\Delta z = \gamma_s\Delta z}$$

Pressure in a fluid increases linearly with depth.

The vertical distance z *is sometimes used as a measure of pressure, and it is* called the *pressure head.*

Variation of Pressure with Depth



Pressure in a liquid at rest increases linearly with distance from the free surface.

$$P = P_{atm} + \rho gh \quad \text{or} \quad P_{gage} = \rho gh$$

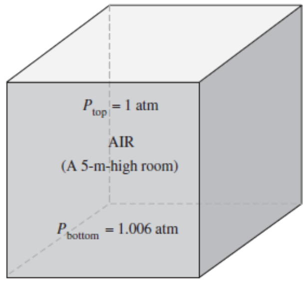
•Liquids are essentially incompressible substances, and thus the variation of density with depth is negligible. This is also the case for gases when the elevation change is not very large.

•The variation of density of liquids or gases with temperature can be significant, also, at great depths such as those encountered in **oceans**, the change in the density of a liquid can be significant because of the compression by the tremendous amount of liquid weight above.

•The gravitational acceleration *g varies from* ***9.807 m/s² at sea level*** *to* **9.764 m/s² at an elevation of 14,000 m** where large passenger planes cruise.

•This is a change of just **0.4 percent in this extreme case.**

•Therefore, ***g can be*** **assumed to be constant with negligible error.**



In a room filled with a gas, the variation of pressure with height is negligible.

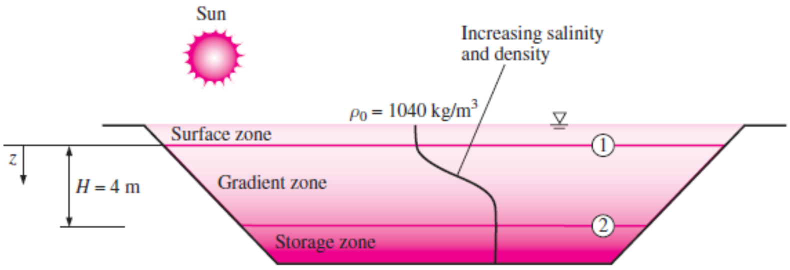
Variation of Pressure with Depth

$$\Delta P = \gamma_s \, \Delta z \qquad \Delta z \to 0 \qquad \frac{dP}{dz} = -\rho g$$

When the variation of density with elevation is known, the pressure difference between points 1 and 2 can be determined by integration to be

$$\Delta P = P_2 - P_1 = -\int_1^2 \rho g \, dz$$

Example: Hydrostatic Pressure in a Solar Pond with Variable Density



$$\rho = \rho_0 \sqrt{1 + \tan^2\left(\frac{\pi}{4}\frac{z}{H}\right)}$$

$P_2 = ?$

Thickness of the surface zone is 0.8 m.

$$P_1 = \rho g h_1 = (1040\ \mathrm{kg/m^3})(9.81\ \mathrm{m/s^2})(0.8\ \mathrm{m})\left(\frac{1\ \mathrm{kN}}{1000\ \mathrm{kg \cdot m/s^2}}\right) = 8.16\ \mathrm{kPa}$$

$$dP = \rho g\, dz \quad \Rightarrow \quad P - P_1 = \int_0^z \rho g\, dz \quad \rightarrow \quad P = P_1 + \int_0^z \rho_0 \sqrt{1 + \tan^2\left(\frac{\pi}{4}\frac{z}{H}\right)} g\, dz$$

$$P = P_1 + \rho_0 g \frac{4H}{\pi} \sinh^{-1}\left(\tan \frac{\pi}{4}\frac{z}{H}\right)$$

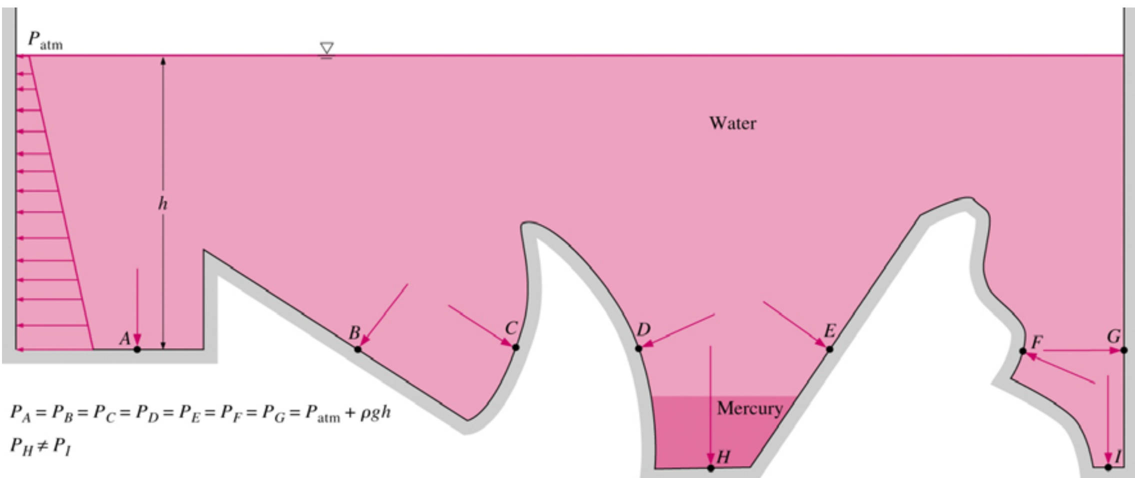
For z=H=4 m $P_2 = 8.16\ \mathrm{kPa} + (1040\ \mathrm{kg/m^3})(9.81\ \mathrm{m/s^2}) \frac{4(4\ \mathrm{m})}{\pi} \sinh^{-1}\left(\tan \frac{\pi}{4}\frac{4}{4}\right)\left(\frac{1\ \mathrm{kN}}{1000\ \mathrm{kg \cdot m/s^2}}\right)$

(gage pressure)

$P_2 = 54$ kPa

Variation of Pressure with Depth

- Pressure in a fluid at rest is independent of the shape of the container.
- Pressure is the same at all points on a horizontal plane in a given fluid.
- The pressure force exerted by the fluid is always normal to the surface at the specified points.



Scuba Diving and Hydrostatic Pressure

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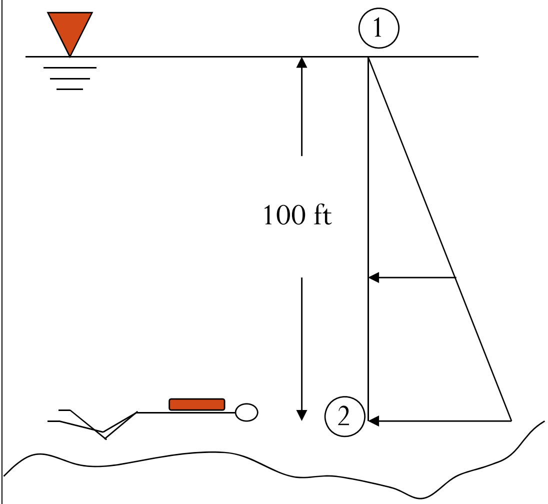


If you hold your breath on ascent, your lung volume would increase by a factor of 4, which would result in embolism and/or death.

- Pressure on diver at 100 ft?

$$P_{gage,2} = \rho g z = \left(998\frac{kg}{m^3}\right)\left(9.81\frac{m}{s^2}\right)(100\,ft)\left(\frac{1m}{3.28\,ft}\right)$$

$$= 298.5kPa\left(\frac{1atm}{101.325kPa}\right) = 2.95atm$$

$$P_{abs,2} = P_{gage,2} + P_{atm} = 2.95atm + 1atm = 3.95atm$$

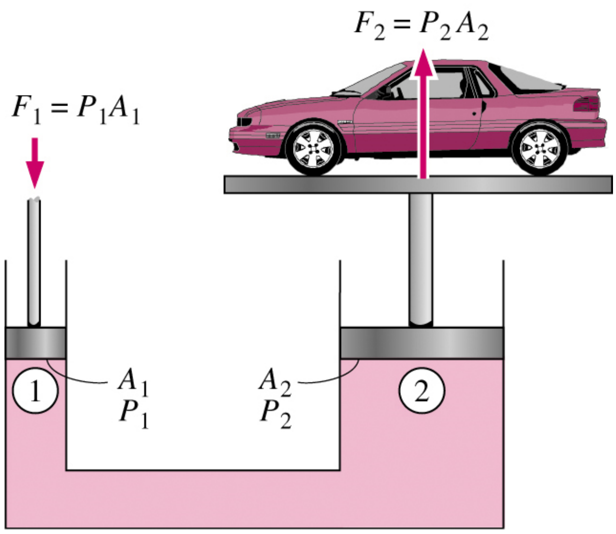
- Danger of emergency ascent?

Boyle's law

$$P_1V_1 = P_2V_2$$

$$\frac{V_1}{V_2} = \frac{P_2}{P_1} = \frac{3.95atm}{1atm} \approx 4$$

Pascal's Law

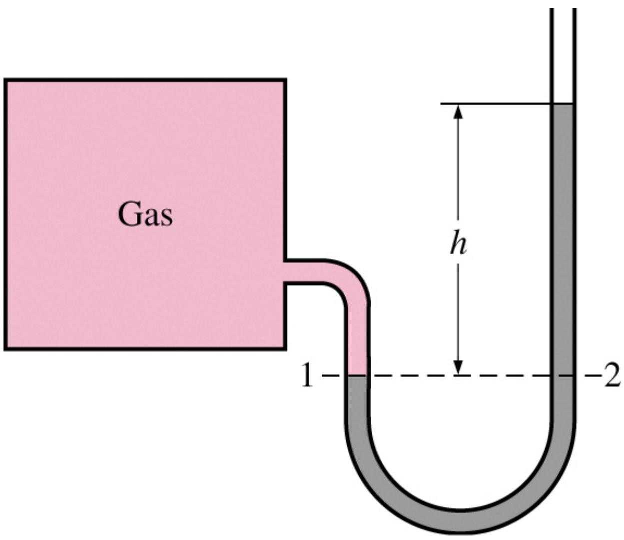


- Pressure applied to a confined fluid increases the pressure throughout by the same amount.
- In picture, pistons are at same height:

$$P_1 = P_2 \rightarrow \frac{F_1}{A_1} = \frac{F_2}{A_2} \rightarrow \frac{F_2}{F_1} = \frac{A_2}{A_1}$$

- Ratio A_2/A_1 is called *ideal mechanical advantage*

The Manometer

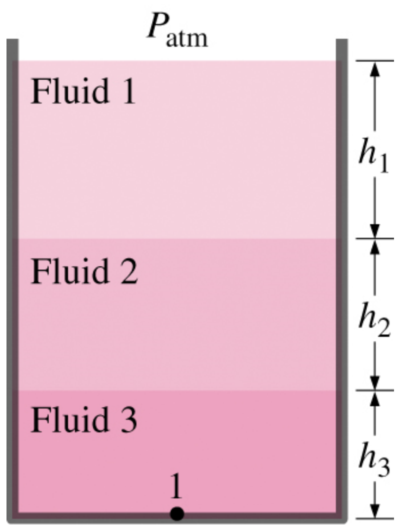


$$P_1 = P_2$$

$$P_2 = P_{atm} + \rho g h$$

- An elevation change of Δz in a fluid at rest corresponds to $\Delta P/\rho g$.
- A device based on this is called a **manometer.**
- A manometer consists of a U-tube containing one or more fluids such as mercury, water, alcohol, or oil.
- Heavy fluids such as mercury are used if large pressure differences are anticipated.
- Two points at the same elevation in a continuous fluid at rest are at the same pressure.

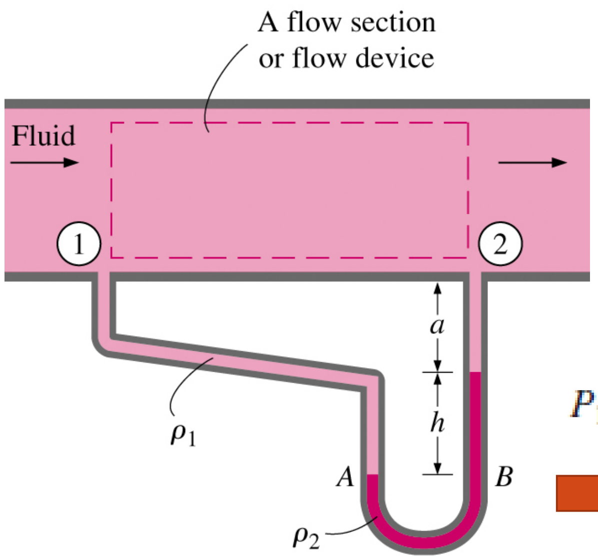
Mutlifluid Manometer



- For multi-fluid systems
  - Pressure change across a fluid column of height h is $\Delta P = \rho g h$.
  - Pressure increases downward, and decreases upward.
  - Two points at the same elevation in a continuous fluid are at the same pressure.

$$P_{atm} + \rho_1 g h_1 + \rho_2 g h_2 + \rho_3 g h_3 = P_1$$

Measuring Pressure Drops



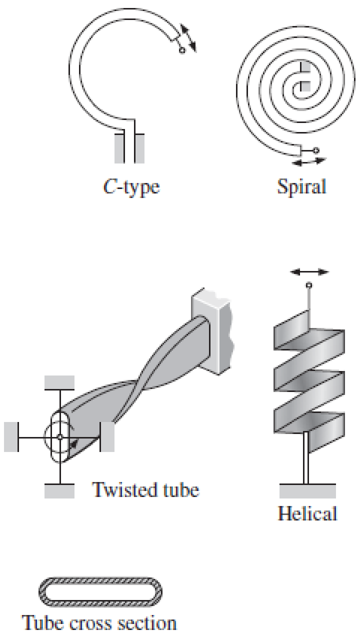
- Manometers are well--suited to measure pressure drops across valves, pipes, heat exchangers, etc.
- Relation for pressure drop P_1-P_2 is obtained by starting at point 1 and adding or subtracting $\rho g h$ terms until we reach point 2.

$$P_1 + \rho_1 g(a + h) - \rho_2 g h - \rho_1 g a = P_2$$

$$P_1 - P_2 = (\rho_2 - \rho_1) g h$$

If fluid in pipe is a gas, **$\rho_2 >> \rho_1$ and $P_1 - P_2 = \rho_2\, g\, h$**

Other Pressure Measurement Devices



Various types of Bourdon tubes used to measure pressure.

- Another type of commonly used mechanical pressure measurement device is the **Bourdon tube, named after the French engineer and inventor Eugene** Bourdon (1808–1884), which consists of a hollow metal tube bent like a hook whose end is closed and connected to a dial indicator needle.
- When the tube is open to the atmosphere, the tube is undeflected, and the needle on the dial at this state is calibrated to read zero (gage pressure). When the fluid inside the tube is pressurized, the tube stretches and moves the needle in proportion to the pressure applied.

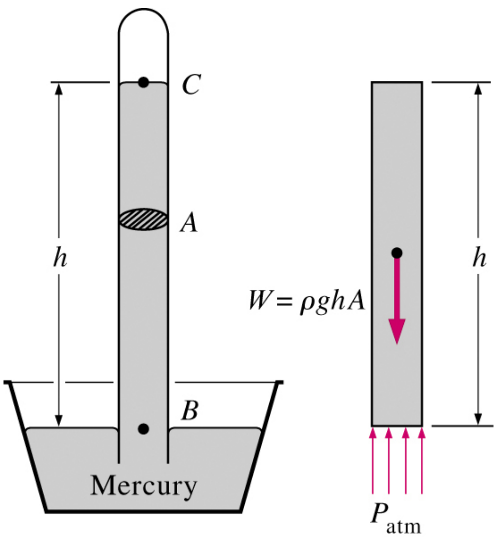
Other Pressure Measurement Devices

- Electronics have made their way into every aspect of life, including pressure measurement devices. Modern pressure sensors, called **pressure transducers,** use various techniques **to convert the pressure effect to an electrical effect such as a change in voltage, resistance, or capacitance**. Pressure transducers are smaller and faster, and they can be more sensitive, reliable, and precise than their mechanical counterparts. They can measure pressures from less than a **millionth of 1 atm to several thousands of atm**.

Other Pressure Measurement Devices

- **Strain-gage pressure transducers** work by having a diaphragm deflect between two chambers open to the pressure inputs.
- **Piezoelectric transducers,** also called solid-state pressure transducers, work on the principle that an electric potential is generated in a crystalline substance when it is subjected to mechanical pressure.
- This phenomenon, first discovered by brothers Pierre and Jacques Curie in 1880, is called the piezoelectric (or press-electric) effect.
- Piezoelectric pressure transducers have a much faster frequency response compared to the diaphragm units and are very suitable for high-pressure applications, but they are generally not as sensitive as the diaphragm-type transducers.

The Barometer



$$P_C + \rho gh = P_{atm}$$

$$P_{atm} = \rho gh$$

- Atmospheric pressure is measured by a device called a **barometer**; thus, atmospheric pressure is often referred to as the *barometric pressure*.
- P_C can be taken to be zero since there is only Hg vapor above point C, and it is very low relative to P_{atm}.
- Change in atmospheric pressure due to elevation has many effects: Cooking, nose bleeds, engine performance, aircraft performance.
- Example: a **2.0-L car engine** will act like **a 1.7-L car engine at 1500 m** altitude (unless it is turbocharged) because of the 15 percent drop in pressure and thus 15 percent drop in the density of air.
- **The lower pressure** and thus **lower density** also **affects lift and drag**:
- airplanes need **a longer runway** at high altitudes to develop the **required lift**
- and airplanes climb to very high altitudes for cruising for **reduced drag** and thus **better fuel efficiency.**

The Barometer

- A frequently used pressure unit is **the *standard atmosphere***, *which is* defined as the pressure produced by **a column of mercury 760 mm** in height at **0°C** (ρ_{Hg} = **13595 kg/m³**) under standard gravitational acceleration **(*g = 9.807 m/s²*).**
- *If* ***water instead of mercury*** *were used to* ***measure the*** **standard atmospheric pressure**, a **water column of about 10.3 m** would be needed.

- 1 mmHg = 1 torr

Elevation (m)	Standard atmospheric pressure, P_{atm} (kPa)
0	101,325
1000	89,98
1610 (Denver, USA)	83,40
2000	79,50
5000	54,05
10000	26,50
20000	5,53

Fluid Statics

- **Fluid Statics** deals with problems associated with fluids at rest.
- In fluid statics, there is no relative motion between adjacent fluid layers.
- Therefore, there is no shear stress in the fluid trying to deform it.
- The only stress in fluid statics is *normal stress*
  - Normal stress is due to pressure
  - Variation of pressure is due only to the weight of the fluid → fluid statics is only relevant in presence of gravity fields.
- Applications: Floating or submerged bodies, water dams and gates, liquid storage tanks, etc.
- The complete description of the resultant hydrostatic force acting on a submerged surface requires the determination of the magnitude, the direction, and the line of action of the force.

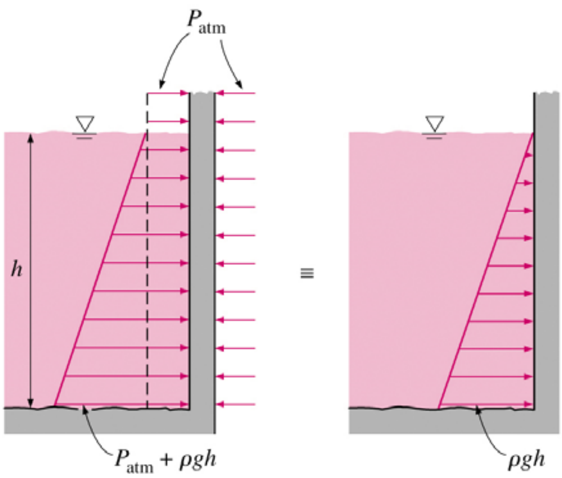
Hoover Dam

Hoover Dam

Hoover Dam

- Example of elevation head *z* converted to velocity head $V^2/2g$. We'll discuss this in more detail in Chapter 5 (Bernoulli equation).

Hydrostatic Forces on Plane Surfaces

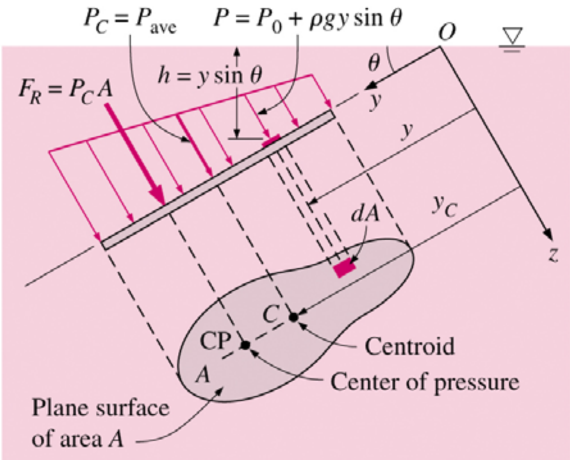


(*a*) P_{atm} considered (*b*) P_{atm} subtracted

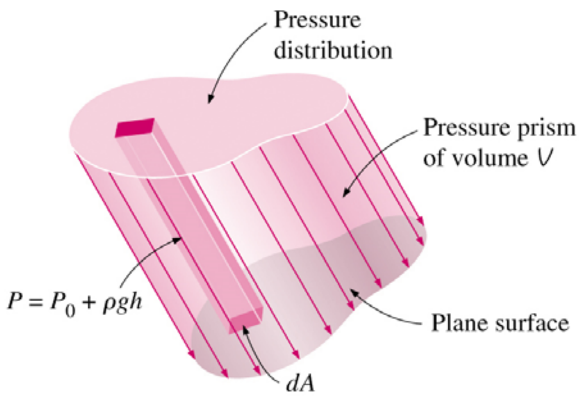
When analyzing hydrostatic forces on submerged surfaces, **the atmospheric pressure can be subtracted** for simplicity **when it acts on both sides of the structure.**

- On a *plane* surface, the hydrostatic forces form a system of parallel forces
- For many applications, magnitude and location of application, which is called **center of pressure**, must be determined.

Resultant Force



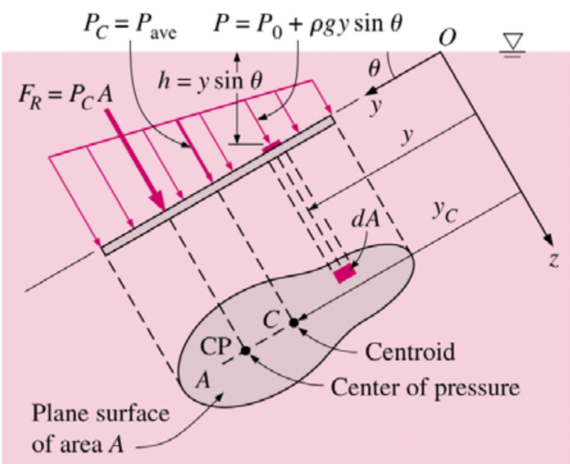
Hydrostatic force on an inclined plane surface completely submerged in a liquid.



$$V = \int dV = \int P\, dA = F_R$$

The magnitude of F_R acting on a plane surface of a completely submerged plate in a homogenous fluid is equal to the product of the pressure P_C at the centroid of the surface and the area A of the surface.

Resultant Force



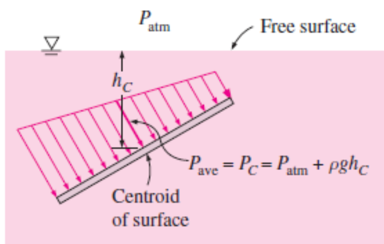
The absolute pressure at any point on the plate is

$$P = P_0 + \rho g h = P_0 + \rho g y \sin\theta$$

$$F_R = \int_A P\, dA = \int_A (P_0 + \rho g y \sin\theta)\, dA = P_0 A + \rho g \sin\theta \int_A y\, dA$$

$$y_C = \frac{1}{A}\int_A y\, dA$$

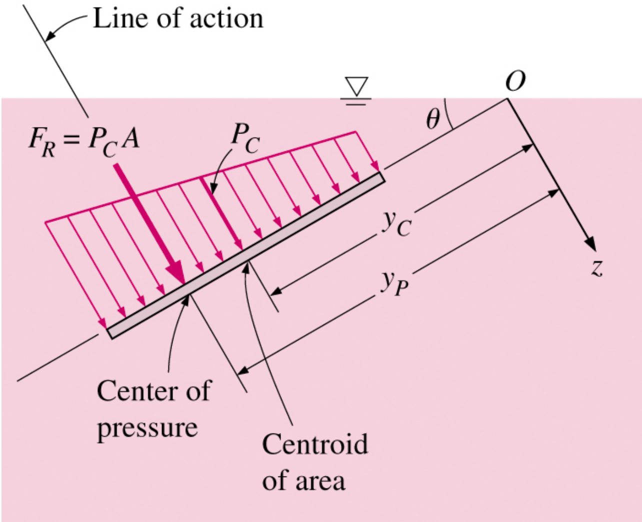
$$F_R = (P_0 + \rho g y_C \sin\theta)A = \underbrace{(P_0 + \rho g h_C)}_{P_C}A = P_C A = P_{ave} A$$



The pressure at the centroid of a surface is equivalent to the *average* pressure on the surface.

Note: The magnitude of the resultant force acting on a plane surface of a completely submerged plate in a homogeneous (constant density) fluid
is equal to the product of the pressure P_C *at the centroid of the surface* and the area A *of the surface*

Center of Pressure



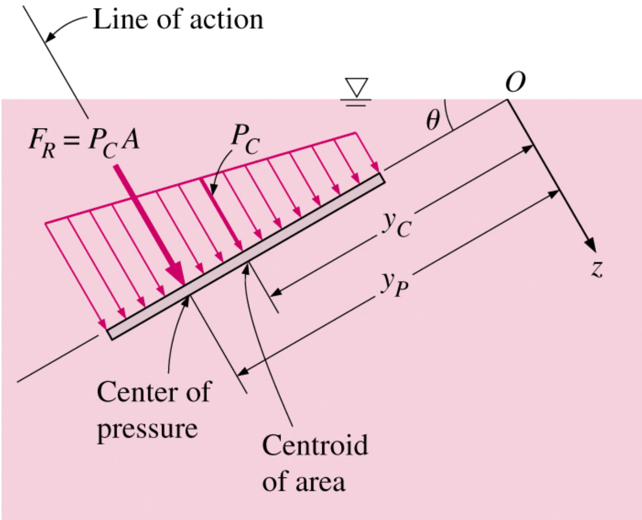
- Line of action of resultant force $F_R=P_CA$ does not pass through the centroid of the surface. In general, it lies underneath where the pressure is higher.
- Vertical location of **Center of Pressure** is determined by equation the moment of the resultant force to the moment of the distributed pressure force.

$$y_P F_R = \int_A yP\,dA = \int_A y(P_0 + \rho g y \sin\theta)\,dA = P_0 \int_A y\,dA + \rho g \sin\theta \int_A y^2\,dA$$

$$y_P F_R = P_0 y_C A + \rho g \sin\theta\, I_{xx,O}$$

$$I_{xx,O} = \int_A y^2\,dA$$ is the *second moment of area (also called* the *area moment of inertia) about the x-axis.*

Center of Pressure



The second moments of area about two parallel axes are related to each other by the *parallel axis theorem, which in this case is expressed as*

$$I_{xx,O} = I_{xx,C} + y_C^2 A$$

$$F_R = (P_0 + \rho g y_C \sin\theta)A$$

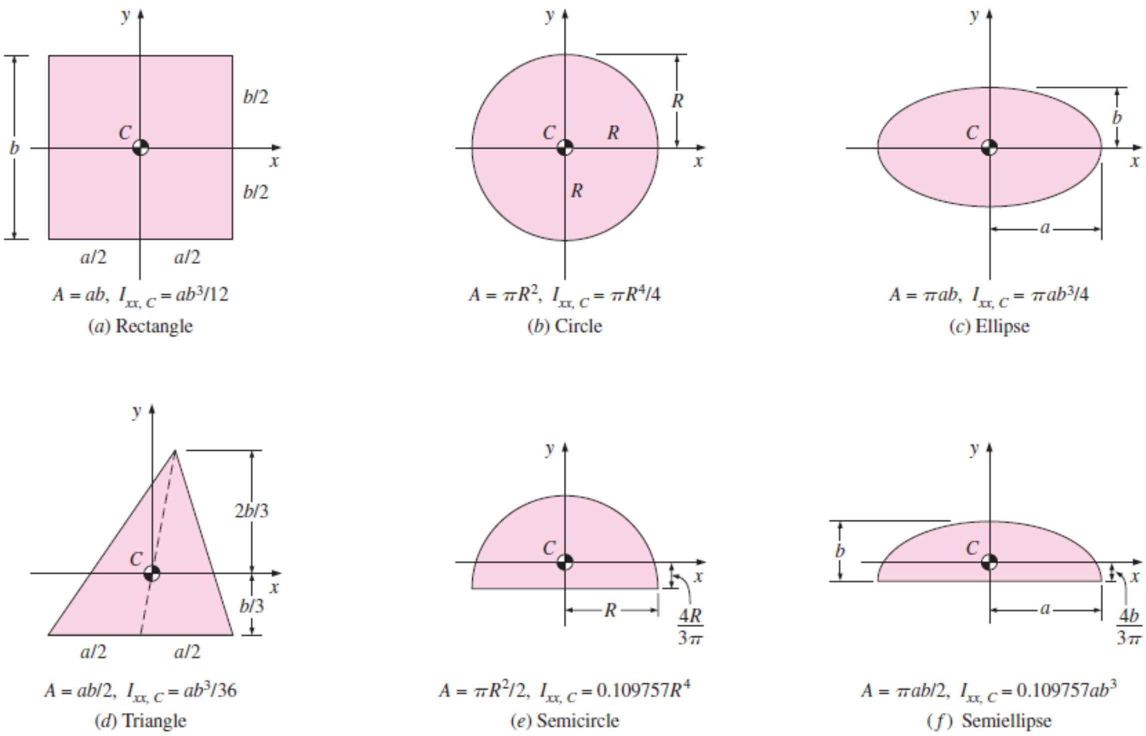
$$y_P F_R = P_0 y_C A + \rho g \sin\theta\, I_{xx,O}$$

$$y_P = y_C + \frac{I_{xx,C}}{[y_C + P_0/(\rho g \sin\theta)]A}$$

For $P_0=0$, *which is usually the case when the atmospheric pressure is* ignored, it simplifies to

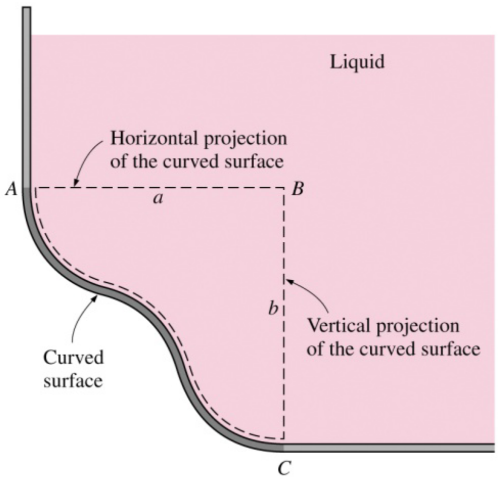
$$y_P = y_C + \frac{I_{xx,C}}{y_C A}$$

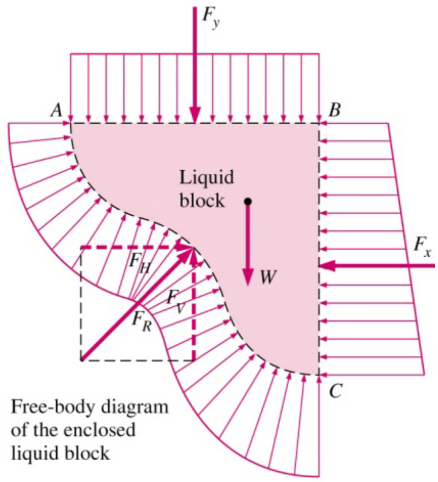
Center of Pressure



The centroid and the centroidal moments of inertia for some common geometries.

Hydrostatic Forces on Curved Surfaces



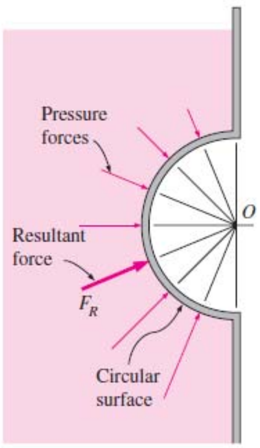


- F_R on a curved surface is more involved since it requires integration of the pressure forces that change direction along the surface.
- Easiest approach: determine horizontal and vertical components F_H and F_V separately.

Hydrostatic Forces on Curved Surfaces

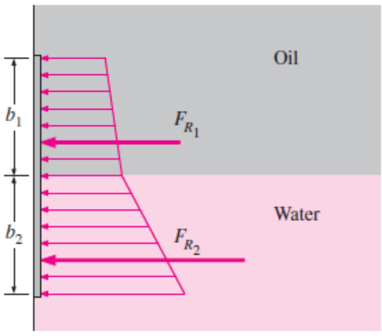
- Horizontal force component on curved surface: $F_H=F_x$. Line of action on vertical plane gives y coordinate of center of pressure on curved surface.
- Vertical force component on curved surface: $F_V=F_y+W$, where W is the weight of the liquid in the enclosed block $W=\rho g V$. x coordinate of the center of pressure is a combination of line of action on horizontal plane (centroid of area) and line of action through volume (centroid of volume).
- Magnitude of force $F_R=(F_H^2+F_V^2)^{1/2}$
- Angle of force is $\alpha = tan^{-1}(F_V/F_H)$

Hydrostatic Forces on Curved Surfaces



When the curved surface is a *circular arc (full circle or any part of it), the* resultant hydrostatic force acting on the surface always passes through the center of the circle. This is because the pressure forces are normal to the surface, and all lines normal to the surface of a circle pass through the center of the circle.

Plane surface in a multilayered fluid:



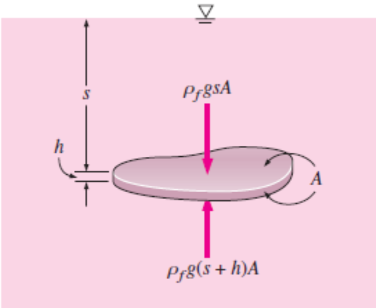
$$F_R = \sum F_{R,i} = \sum P_{C,i} A_i$$

$$P_{C,i} = P_0 + \rho_i g h_{C,i}$$

Buoyancy and Stability

- It is a common experience that an object feels lighter and weighs less in a liquid than it does in air.
- Also, objects made of wood or other light materials float on water.
- These and other observations suggest that a fluid exerts an upward force on a body immersed in it. This force that tends to lift the body is called the **buoyant force and is denoted** by F_B
- Buoyancy is due to the fluid displaced by a body. $F_B = \rho_f g V$.
- **Archimedes principal** : The buoyant force acting on a body immersed in a fluid is equal to the weight of the fluid displaced by the body, and it acts upward through the **centroid** of the displaced volume.

Buoyancy and Stability



A flat plate of uniform thickness h submerged in a liquid parallel to the free surface.

•The buoyant force is caused by the increase of pressure in a fluid with depth.

•The difference between these two forces is a net upward force, which is the *buoyant force.*

$$F_B = F_{\text{bottom}} - F_{\text{top}} = \rho_f g(s+h)A - \rho_f gsA = \rho_f ghA = \rho_f gV$$

•The relation $\rho_f gV$ *is simply the* weight of the liquid whose volume is equal to the volume of the plate.

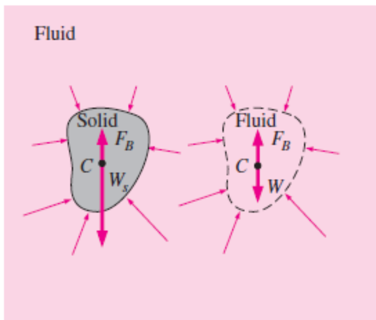
•Thus, we conclude that *the buoyant force acting on the plate is equal to the weight of the liquid displaced by the plate.*

•The buoyant force is independent of the distance of the body from the free surface. It is also independent of the density of the solid body.

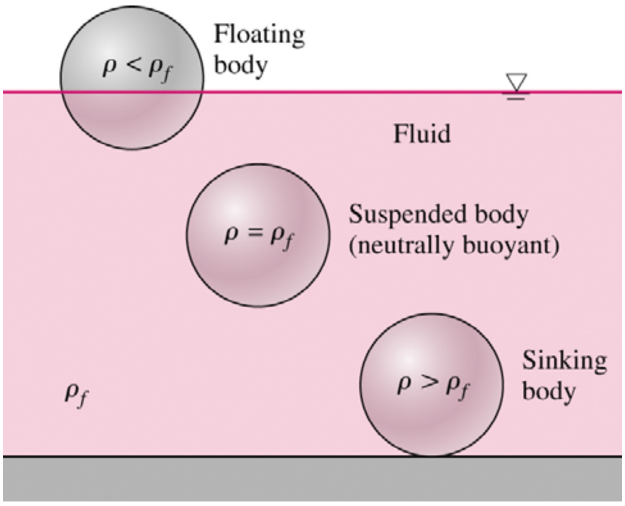
Buoyancy and Stability

•The weight and the buoyant force must have the same line of action to have a zero moment. This is known as **Archimedes' principle, after the Greek** mathematician Archimedes (287–212 BC)

•The buoyant force F_B *acts upward through the* centroid C *of the displaced volume*



Buoyancy and Stability



- Buoyancy force F_B is equal only to the displaced volume

  $$\rho_f g V_{displaced}$$

- Three scenarios possible
  1. $\rho_{body} < \rho_{fluid}$: Floating body
  2. $\rho_{body} = \rho_{fluid}$: Neutrally buoyant
  3. $\rho_{body} > \rho_{fluid}$: Sinking body

For *floating bodies, the weight of the entire body must be equal to the* bouyant force, which is the weight of the fluid whose volume is equal to the volume of the submerged portion of the floating body.

$$F_B = W \rightarrow \rho_f g V_{sub} = \rho_{ave,\ body} g V_{total} \rightarrow \frac{V_{sub}}{V_{total}} = \frac{\rho_{ave,\ body}}{\rho_f}$$

Buoyancy and Stability

- We might think that the buoyant force exerted by gases such as air is negligible. This is certainly the case in general, but there are significant exceptions. For example, the volume of a person is about 0.1 m^3, and taking the density of air to be 1.2 kg/m^3, the buoyant force exerted by air on the person is

  $$F_B = \rho_f g V = (1.2\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(0.1\ \text{m}^3) \cong 1.2\ \text{N}$$

  The weight of an 80-kg person is 80 x 9.81 = 788 N. Therefore, ignoring the buoyancy in this case results in an error in weight of just 0.15 percent.
- But **the buoyancy effects** in gases dominate some important natural phenomena such as the rise of warm air in a cooler environment and thus the onset **of natural convection currents**, the **rise of hot-air or helium balloons, and air movements in the atmosphere.**
- A helium balloon, for example, rises as a result of the buoyancy effect until it reaches an altitude where the density of air (which decreases with altitude) equals the density of helium in the balloon.

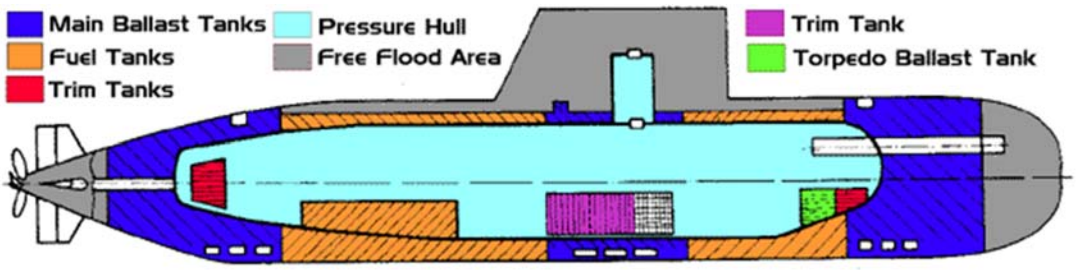
Example: Floating Drydock

Auxiliary Floating Dry Dock Resolute (AFDM-10) partially submerged

Submarine undergoing repair work on board the AFDM-10

Using buoyancy, a submarine with a displacement of 6,000 tons can be lifted!

Example: Submarine Buoyancy and Ballast



- Submarines use both static and dynamic depth control. Static control uses ballast tanks between the pressure hull and the outer hull. Dynamic control uses the bow and stern planes to generate trim forces.

Example: Submarine Buoyancy and Ballast

Normal surface trim

SSN 711 nose down after accident
which damaged fore ballast tanks

Example: Submarine Buoyancy and Ballast

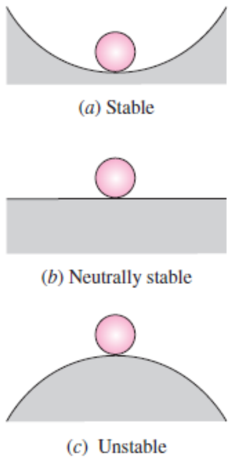
Damage to SSN 711 (USS San Francisco) after running aground on 8 January 2005.

Example: Submarine Buoyancy and Ballast

Ballast Control Panel: Important station for controlling depth of submarine

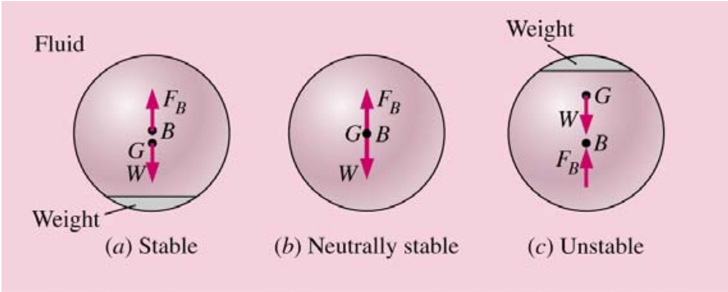
Stability of Immersed and Floating Bodies

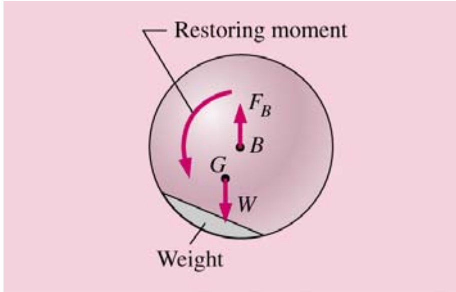
- An important application of the buoyancy concept is the assessment of the stability of immersed and floating bodies with no external attachments. This topic is of great importance in the design of ships and submarines



Stability is easily understood by analyzing a ball on the floor.

Stability of Immersed Bodies



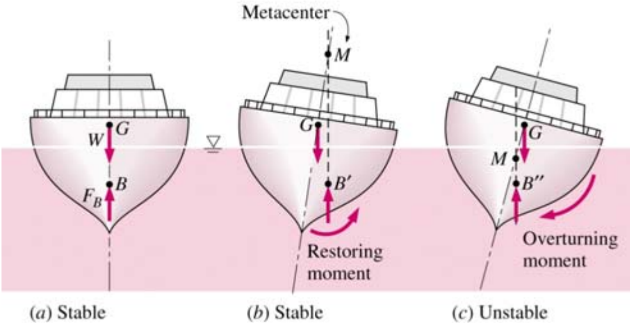


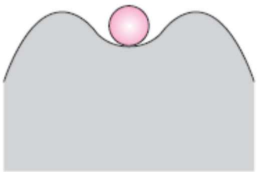
- Rotational stability of immersed bodies depends upon relative location of *center of gravity* G and *center of buoyancy* B.
  - G below B: stable
  - G above B: unstable
  - G coincides with B: neutrally stable.

Stability of Immersed Bodies

- A stable design for a submarine calls for the engines and the cabins for the crew to be located at the lower half in order to shift the weight to the bottom as much as possible.
- Hot-air or helium balloons (which can be viewed as being immersed in air) are also stable since the cage that carries the load is at the bottom.

Stability of Floating Bodies



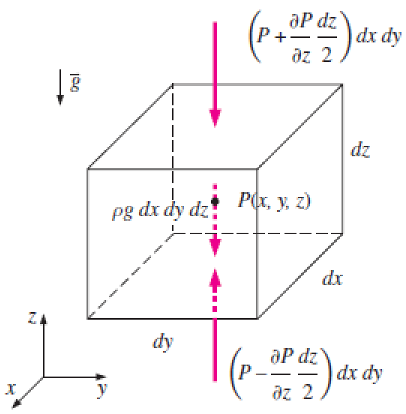


A ball in a trough between two hills is stable for small disturbances, but unstable for large disturbances.

- If body is bottom heavy (*G* lower than *B*), it is always stable.
- Floating bodies can be stable when *G* is higher than *B* due to shift in location of center buoyancy and creation of restoring moment.
- **Measure of stability** is the **metacentric height G*M***. If *GM*>0, ship is stable.

The metacenter may be considered to be a fixed point for most hull shapes for small rolling angles up to about 20°.

Rigid-Body Motion



The surface and body forces acting on a differential fluid element in the vertical direction.

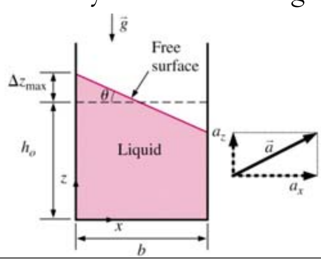
The differential fluid element behaves like a *rigid body, Newton's second law of motion*

$$\delta\vec{F} = \delta m \cdot \vec{a}$$

$$\delta m = \rho\, dV = \rho\, dx\, dy\, dz$$

Forces: 1) body force (gravity force)
2) surface forces (pressure forces)

➢ Pressure represents the compressive force applied on the fluid element by the surrounding fluid and is always directed to the surface.



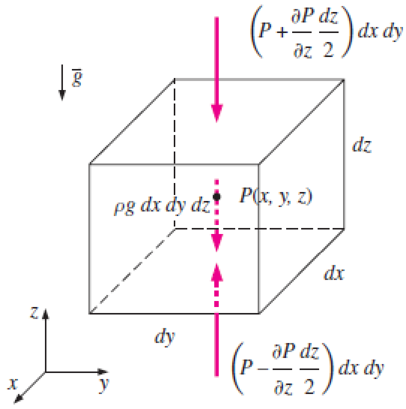
Rigid-Body Motion

- The net surface force acting on the element in the *z-direction is the difference* between the pressure forces acting on the bottom and top faces

$$\delta F_{S,z} = \left(P - \frac{\partial P}{\partial z}\frac{dz}{2}\right) dx\, dy - \left(P + \frac{\partial P}{\partial z}\frac{dz}{2}\right) dx\, dy = -\frac{\partial P}{\partial z}\, dx\, dy\, dz$$

Similarly, the net surface forces in the *x- and y-directions are*

$$\delta F_{S,x} = -\frac{\partial P}{\partial x}\, dx\, dy\, dz \quad \text{and} \quad \delta F_{S,y} = -\frac{\partial P}{\partial y}\, dx\, dy\, dz$$



The surface and body forces acting on a differential fluid element in the vertical direction.

The surface force acting on the entire element:

$$\delta \vec{F}_S = \delta F_{S,x}\vec{i} + \delta F_{S,y}\vec{j} + \delta F_{S,z}\vec{k}$$

$$= -\left(\frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k}\right) dx\, dy\, dz = -\vec{\nabla} P\, dx\, dy\, dz$$

Rigid-Body Motion

Pressure gradient : $\vec{\nabla} P = \frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k}$

$\vec{\nabla}$ or **"del"** is a vector operator that is used to express the gradients of a scalar function compactly in vector form.

The weight of the element acting in the negative *z-direction*

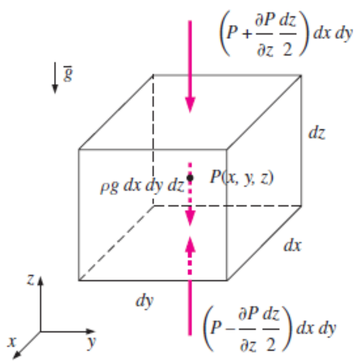
$$\delta \vec{F}_{B,z} = -g\delta m\vec{k} = -\rho g\, dx\, dy\, dz\vec{k}$$

The total force acting on the element becomes

$$\delta \vec{F} = \delta \vec{F}_S + \delta \vec{F}_B = -(\vec{\nabla} P + \rho g\vec{k})\, dx\, dy\, dz$$

The general **equation of motion for a fluid that** acts as a rigid body (no shear stresses)

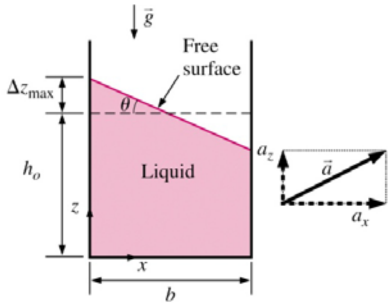
$$\delta \vec{F} = \delta m \cdot \vec{a} \qquad \delta m = \rho dxdydz$$



Rigid-body motion of fluids: $\boxed{\vec{\nabla} P + \rho g\vec{k} = -\rho\vec{a}}$

The surface and body forces acting on a differential fluid element in the vertical direction.

Rigid-Body Motion



Rigid-body motion of fluids: $$\vec{\nabla}P + \rho g\vec{k} = -\rho\vec{a}$$

Explicitly: $$\frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k} + \rho g\vec{k} = -\rho(a_x\vec{i} + a_y\vec{j} + a_z\vec{k})$$

In scalar form in the three orthogonal directions

$$\frac{\partial P}{\partial x} = -\rho a_x, \quad \frac{\partial P}{\partial y} = -\rho a_y, \quad \text{and} \quad \frac{\partial P}{\partial z} = -\rho(g + a_z)$$

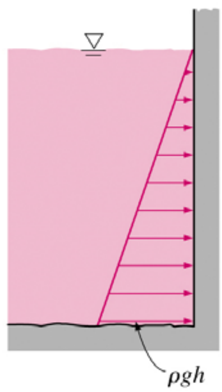
a_x, a_y, and a_z are accelerations in the x-, y-, and z-directions, respectively.

Special Case 1: Fluids at Rest

General case: $$\frac{\partial P}{\partial x} = -\rho a_x, \quad \frac{\partial P}{\partial y} = -\rho a_y, \quad \text{and} \quad \frac{\partial P}{\partial z} = -\rho(g + a_z)$$

Fluids at rest: $$\frac{\partial P}{\partial x} = 0, \quad \frac{\partial P}{\partial y} = 0, \quad \text{and} \quad \frac{dP}{dz} = -\rho g$$

These relations are applicable for both compressible and incompressible fluids.



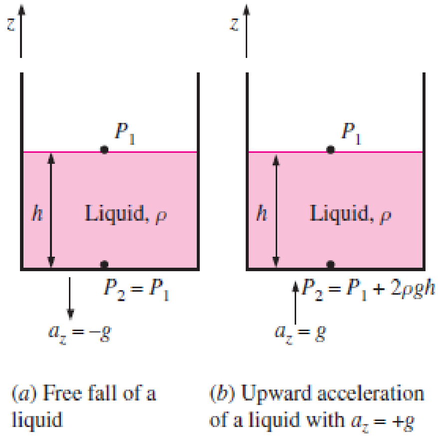
Special Case 2: Free Fall of a Fluid Body

General case: $\frac{\partial P}{\partial x} = -\rho a_x, \quad \frac{\partial P}{\partial y} = -\rho a_y, \quad \text{and} \quad \frac{\partial P}{\partial z} = -\rho(g + a_z)$

a_x , $a_y = 0$ *and* $a_z = -g$

Free-falling fluids: $\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = \frac{\partial P}{\partial z} = 0 \quad \rightarrow \quad P = \text{constant}$

(Actually, the gage pressure is slightly above zero due to surface tension, which holds the drop intact.)



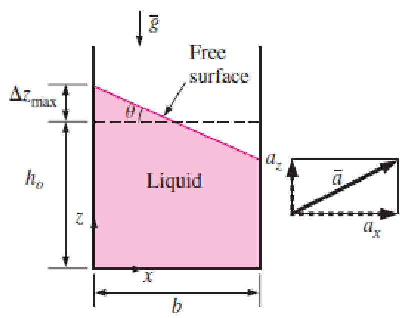
The effect of acceleration on the pressure of a liquid during free fall and upward acceleration.

Acceleration on a Straight Path

$$\frac{\partial P}{\partial x} = -\rho a_x, \quad \frac{\partial P}{\partial y} = 0, \quad \text{and} \quad \frac{\partial P}{\partial z} = -\rho(g + a_z)$$

Total derivative of P(x,y) : $dP = \frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy$

$$dP = -\rho a_x\, dx - \rho(g + a_z)\, dz$$



Rigid-body motion of a liquid in a linearly accelerating tank.

For ***ρ=constant***, the pressure difference between two points 1 and 2 in the fluid.

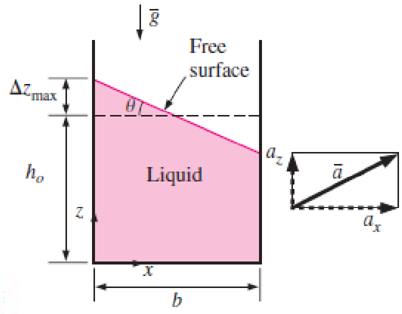
$$P_2 - P_1 = -\rho a_x(x_2 - x_1) - \rho(g + a_z)(z_2 - z_1)$$

Acceleration on a Straight Path

$$P_2 - P_1 = -\rho a_x(x_2 - x_1) - \rho(g + a_z)(z_2 - z_1)$$

Point 1: ($x_1=0$, $y_1=0$) $P(0,0)=P_o$
Point 2: any x and y $P(x,y)=P$

Pressure variation: $P = P_0 - \rho a_x x - \rho(g + a_z)z$



Rigid-body motion of a liquid in a linearly accelerating tank.

- $P_1 = P_2$

Vertical rise of surface: $$\Delta z_s = z_{s2} - z_{s1} = -\frac{a_x}{g + a_z}(x_2 - x_1)$$

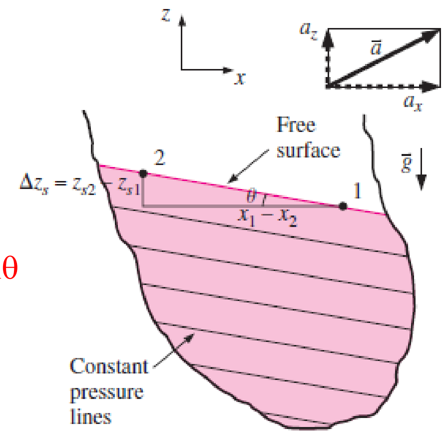
z_s is the z-coordinate of the liquid's free surface

Acceleration on a Straight Path

The equation for surfaces of constant pressure, called **isobars, is obtained** setting *dP =0 and replacing z by z_{isobar}, which is the z-coordinate (the vertical* distance) of the surface as a function of *x*.

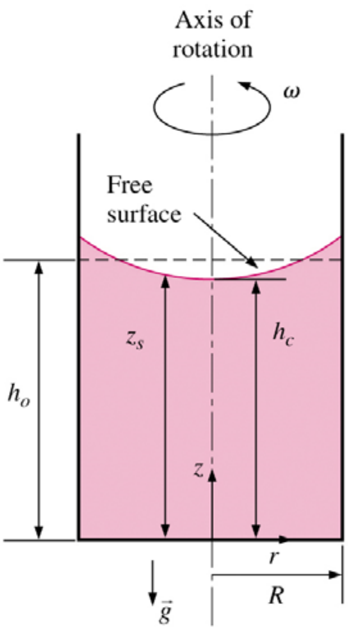
$$dP = -\rho a_x\, dx - \rho(g + a_z)\, dz = 0$$

Surfaces of constant pressure: $$\frac{dz_{\text{isobar}}}{dx} = -\frac{a_x}{g + a_z} = \text{constant} = -\tan\theta$$



Lines of constant pressure (which are the projections of the surfaces of constant pressure on the *xz*-plane) in a linearly accelerating liquid, and the vertical rise.

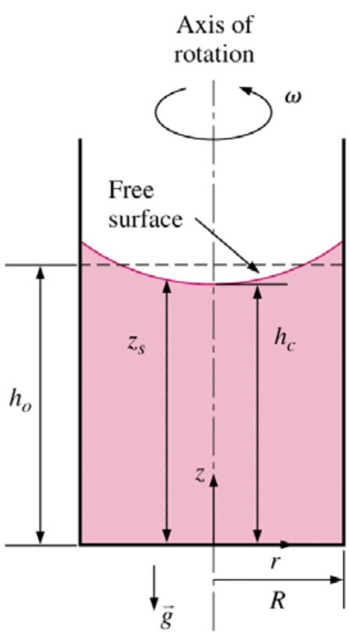
Rotation in a Cylindrical Container



➢We know from experience that when a glass filled with water is rotated about its axis, the fluid is forced outward as a result of the so-called centrifugal force, and the free surface of the liquid becomes concave. This is known as the ***forced vortex motion.***

➢ There is no deformation, and thus there can be no shear stress, and every fluid particle in the container moves with the same angular velocity.

Rotation in a Cylindrical Container



➢The **centripetal acceleration** of a fluid particle rotating with a **constant angular velocity of *ω*** at a distance *r from the* axis of rotation is ***rω*²** *and is directed radially toward the axis of rotation* (negative *r-direction).*

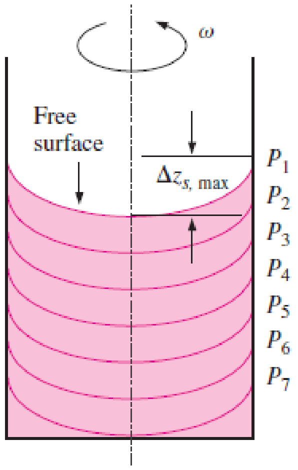
$$a_r = -r\omega^2$$

$$\frac{\partial P}{\partial r} = -\rho a_r = \rho r\omega^2 \quad , \quad \frac{\partial P}{\partial \theta} = -\rho a_\theta = 0 \quad , \quad \frac{\partial P}{\partial z} = -\rho(g + a_z) = -\rho g$$

Total derivative of P(r,z) : $dP = \frac{\partial P}{\partial r}dr + \frac{\partial P}{\partial z}dz$

$$dP = \rho r\omega^2\, dr - \rho g\, dz$$

Rotation in a Cylindrical Container



Surfaces of constant pressure in a rotating liquid.

$$dP = \rho r\omega^2\, dr - \rho g\, dz$$

$$dP = 0 \Rightarrow \frac{dz_{\text{isobar}}}{dr} = \frac{r\omega^2}{g}$$

Surfaces of constant pressure: $z_{\text{isobar}} = \frac{\omega^2}{2g} r^2 + C_1$

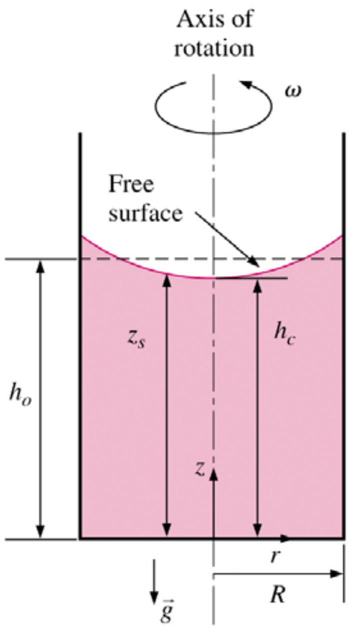
Thus we conclude that the surfaces of constant pressure, including the free surface, are *paraboloids of revolution*

$$z_{isobar}(r=0) = C_1 = h_c \Rightarrow$$

The equation for the free surface

$$z_s = \frac{\omega^2}{2g} r^2 + h_c$$

Rotation in a Cylindrical Container



The equation of the free surface

$$V = \int_{r=0}^{R} 2\pi z_s r\, dr = 2\pi \int_{r=0}^{R} \left(\frac{\omega^2}{2g} r^2 + h_c\right) r\, dr = \pi R^2 \left(\frac{\omega^2 R^2}{4g} + h_c\right)$$

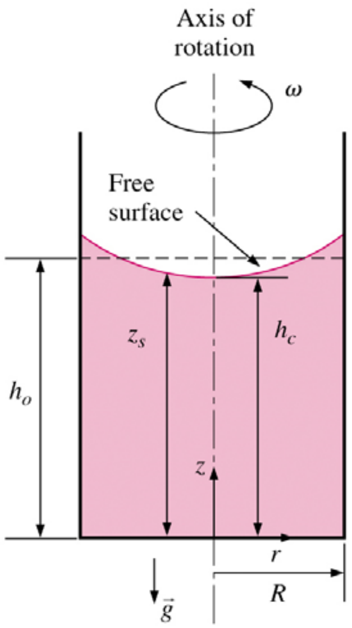
$$V = \pi R^2 h_0 \Rightarrow h_c = h_0 - \frac{\omega^2 R^2}{4g}$$

$$z_s = \frac{\omega^2}{2g} r^2 + h_c \Rightarrow$$

Free surface: $z_s = h_0 - \frac{\omega^2}{4g}(R^2 - 2r^2)$

Maximum height difference: $\Delta z_{s,\,\max} = z_s(R) - z_s(0) = \frac{\omega^2}{2g} R^2$

Rotation in a Cylindrical Container



$$dP = \rho r \omega^2 \, dr - \rho g \, dz$$

$$P_2 - P_1 = \frac{\rho\omega^2}{2}(r_2^2 - r_1^2) - \rho g(z_2 - z_1)$$

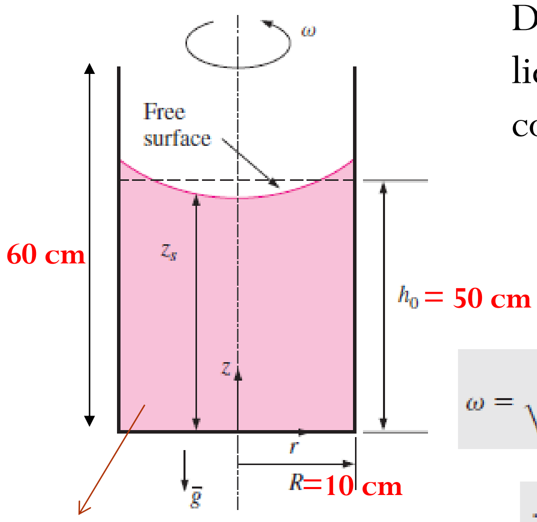
Point 1: $(r_1=0 \, , z_1=0)$ $P(0,0)=P_o$

Point 2: any r and z $P(r,z)=P$

Pressure variation: $P = P_0 + \dfrac{\rho\omega^2}{2} r^2 - \rho g z$

In any horizontal plane, the pressure difference between the center and edge of the container of radius R *is* $\Delta P = \rho\omega^2 R^2/2$

EXAMPLE Rising of a Liquid During Rotation



ρ = 850 kg/m³

Determine the rotational speed at which the liquid will start spilling from the edges of the container.

$$z_s = h_0 - \frac{\omega^2}{4g}(R^2 - 2r^2) \;\Rightarrow\; z_s(R) = h_0 + \frac{\omega^2 R^2}{4g}$$

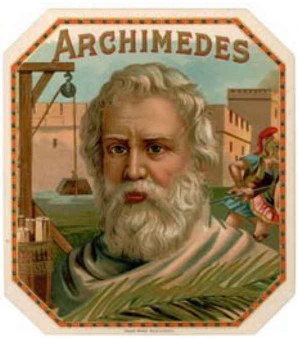
$$\omega = \sqrt{\frac{4g[z_s(R) - h_0]}{R^2}} = \sqrt{\frac{4(9.81 \text{ m/s}^2)[(0.6 - 0.5) \text{ m}]}{(0.1 \text{ m})^2}} = 19.8 \text{ rad/s}$$

$$\dot{n} = \frac{\omega}{2\pi} = \frac{19.8 \text{ rad/s}}{2\pi \text{ rad/rev}}\left(\frac{60 \text{ s}}{1 \text{ min}}\right) = 189 \text{ rpm}$$

The analysis is valid for any liquid since the result is independent of density or any other fluid property

Examples of Archimedes Principle

The Golden Crown of Hiero II, King of Syracuse



- Archimedes, 287-212 B.C.
- Hiero, 306-215 B.C.
- Hiero learned of a rumor where the goldsmith replaced some of the gold in his crown with silver. Hiero asked Archimedes to determine whether the crown was pure gold.
- Archimedes had to develop a nondestructive testing method

The Golden Crown of Hiero II, King of Syracuse

- The weight of the crown and nugget are the same in air: $W_c = \rho_c V_c = W_n = \rho_n V_n$.
- If the crown is pure gold, $\rho_c = \rho_n$ which means that the volumes must be the same, $V_c = V_n$.
- In water, the buoyancy force is $B = \rho_{H2O} V$.
- If the scale becomes unbalanced, this implies that the $V_c \neq V_n$, which in turn means that the $\rho_c \neq \rho_n$
- Goldsmith was shown to be a fraud!

Hydrostatic Bodyfat Testing

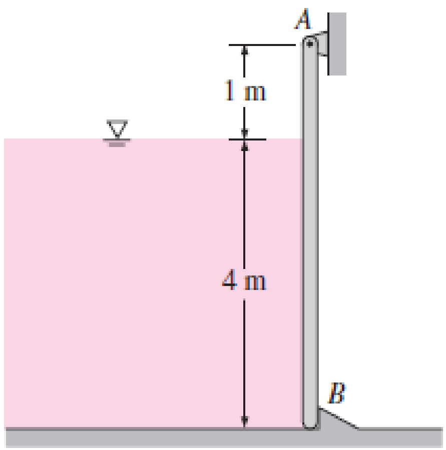
- What is the best way to measure body fat?
- Hydrostatic Bodyfat Testing using Archimedes Principle!
- Process
  - Measure body weight $W = \rho_{body} V$
  - Get in tank, expel all air, and measure apparent weight W_a
  - Buoyancy force $B = W - W_a = \rho_{H2O} V$. This permits computation of body volume.
  - Body density can be computed $\rho_{body} = W/V$.
  - Body fat can be computed from formulas.

Hydrostatic Bodyfat Testing in Air?

- Same methodology as Hydrostatic testing in water.
- What are the ramifications of using air?
  - Density of air is 1/1000th of water.
  - Temperature dependence of air.
  - Measurement of small volumes.
  - Used by NCAA Wrestling (there is a BodPod on PSU campus).

Example

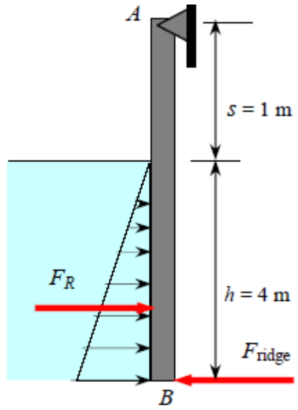
A 4-m-high, 5-m-wide rectangular plate blocks the end of a 4-m-deep freshwater channel, as shown in Figure. The plate is hinged about a horizontal axis along its upper edge through a point *A and is restrained from opening* by a fixed ridge at point *B. Determine the force exerted on* the plate by the ridge.



Example

Assumptions Atmospheric pressure acts on both sides of the plate, and thus it can be ignored in calculations for convenience.

Analysis The average pressure on a surface is the pressure at the centroid (midpoint) of the surface, and is determined to be



$$P_{avg} = P_C = \rho g h_C = \rho g(h/2)$$
$$= (1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(4/2 \text{ m})\left(\frac{1 \text{ kN}}{1000 \text{ kg}\cdot\text{m/s}^2}\right) = 19.62 \text{ kN/m}^2$$

Then the resultant hydrostatic force on each wall becomes

$$F_R = P_{avg}A = (19.62 \text{ kN/m}^2)(4 \text{ m} \times 5 \text{ m}) = 392 \text{ kN}$$

The line of action of the force passes through the pressure center, which is $2h/3$ from the free surface,

$$y_P = \frac{2h}{3} = \frac{2 \times (4 \text{ m})}{3} = 2.667 \text{ m}$$

Taking the moment about point A and setting it equal to zero gives

$$\sum M_A = 0 \quad \rightarrow \quad F_R(s + y_P) = F_{\text{ridge}}\overline{AB}$$

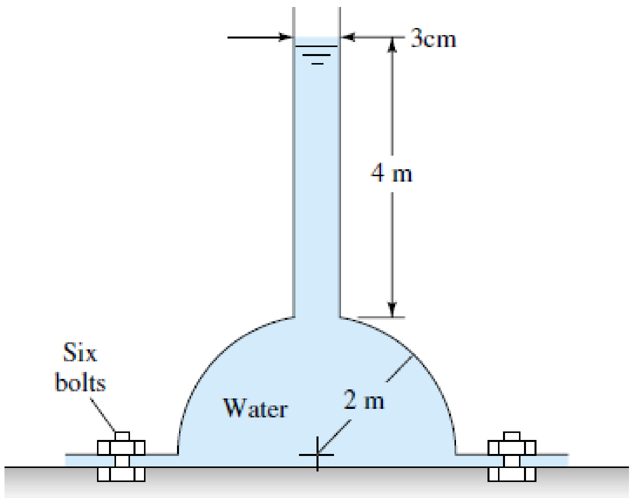
Solving for F_{ridge} and substituting, the reaction force is determined to be

$$F_{\text{ridge}} = \frac{s + y_P}{\overline{AB}}F_R = \frac{(1 + 2.667)\text{ m}}{5 \text{ m}}(392 \text{ kN}) = \mathbf{288\ kN}$$

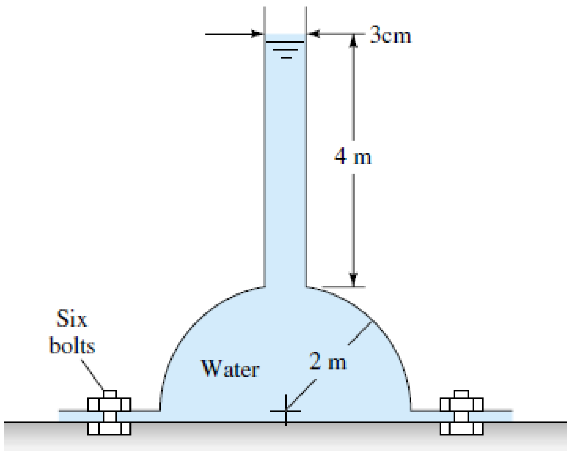
Discussion The difference between F_R and F_{ridge} is the force acting on the hinge at point A.

Example

The hemispherical dome in Figure weighs 30 kN and is filled with water and attached to the floor by six equally spaced bolts. What is the force in each bolt required to hold down the dome?



Example



Solution: Assuming no leakage, the hydrostatic force required equals the *weight of missing water*, that is, the water in a 2-m-diameter cylinder, 6 m high, minus the hemisphere and the small pipe:

$$F_{total} = W_{2\text{-m-cylinder}} - W_{2\text{-m-hemisphere}} - W_{3\text{-cm-pipe}}$$

$$= (9790)\pi(2)^2(6) - (9790)(2\pi/3)(2)^3 - (9790)(\pi/4)(0.03)^2(4)$$

$$= 738149 - 164033 - 28 = 574088 \text{ N}$$

The dome material helps with 30 kN of weight, thus the bolts must supply 574088-30000 or 544088 N. Tthe force in each of 6 bolts is 544088/6 or $F_{bolt} \approx$ **90700 N** *Ans*.

Summary

$$P_{\text{gage}} = P_{\text{abs}} - P_{\text{atm}}$$

$$P_{\text{vac}} = P_{\text{atm}} - P_{\text{abs}}$$

- The variation of pressure with elevation in a fluid at rest is given by

$$\frac{dP}{dz} = -\rho g$$

- When the density of the fluid is constant, the pressure difference across a fluid layer of thickness *z is*

$$\Delta P = P_2 - P_1 = \rho g \, \Delta z$$

- The absolute and gage pressures in a static liquid open to the atmosphere at a depth *h from the free surface are*

$$P = P_{\text{atm}} + \rho g h \qquad \text{and} \qquad P_{\text{gage}} = \rho g h$$

Summary

- The magnitude of the resultant force acting on a plane surface of a completely submerged plate in a homogeneous fluid is

$$F_R = (P_0 + \rho g h_C)A = P_C A = P_{\text{ave}} A$$

- The vertical location of the line of action of the resultant force is given by

$$y_P = y_C + \frac{I_{xx,\,C}}{[y_C + P_0/(\rho g \sin\theta)]A}$$

$I_{xx,\,C}$: *the second moment of area about the x-axis* passing through the centroid of the area

Summary

- A fluid exerts an upward force on a body immersed in it. This force is called the *buoyant force and is expressed as*

$$F_B = \rho_f g V$$

V is the volume of the body in fluid. This is known as Archimedes' principle and is expressed as: the buoyant force acting on a body immersed in a fluid is equal to the weight of the fluid displaced by the body; it acts upward through the centroid of the displaced volume.

- The general *equation of motion for a fluid that acts as a* rigid body is

$$\vec{\nabla} P + \rho g \vec{k} = -\rho \vec{a}$$

$$\frac{\partial P}{\partial x} = -\rho a_x, \qquad \frac{\partial P}{\partial y} = -\rho a_y, \qquad \text{and} \qquad \frac{\partial P}{\partial z} = -\rho(g + a_z)$$

Summary

- During rigid-body motion of a liquid in a *rotating cylinder,* the variation of pressure in the liquid is expressed as

$$P = P_0 + \frac{\rho\omega^2}{2} r^2 - \rho g z$$

P_0 is the pressure at the origin ($r = 0, z = 0$)

Homework

9,24,29,26,32,34,40,42,46,47,51,61E,64,66,70,73,75,85,88
97E,102,105,108E,111,119,121,123,129,133,137,141E,142,144,
143,146,148